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## A METHOD FOR OBTAINING ANALYTICAL SOLUTIONS TO THE EQUATION FOR WIND-DRIVEN CIRCULATION IN A SHALLOW SEA OR LAKE

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# A METHOD FOR OBTAINING ANALYTICAL SOLUTIONS TO THE EQUATION FOR WIND-DRIVEN CIRCULATION IN A SHALLOW SEA OR LAKE

by Marvin E. Goldstein, Willis H. Braun, and Richard T. Gedney

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## SUMMARY

A technique for obtaining solutions to a certain class of partial differential equations is introduced. This class of equations includes Welander's equation for the wind-driven circulation in shallow seas and lakes for a large class of bottom topographies. The technique, together with some results based on conformal mapping, is used to reduce the problems of finding solutions to Welander's equation for a closed body of water to the problem of solving an ordinary differential equation. To illustrate the principles involved, the method is used to obtain the complete analytical solution for an elliptically shaped body of water with a particular choice of bottom topography when the depth is greater than one-half the Ekman friction depth.

## INTRODUCTION

Lake currents are very important in determining what chemical and biological changes will take place in a lake, since they control the distributions of sediment and pollution which have been introduced into a lake. However, it is a major and expensive task to measure the lake currents in large lakes at very many locations. It is, therefore, important to be able to predict these currents analytically. Welander (ref. 1) derived the basic partial differential equation which describes the wind-driven circulation in a shallow sea or lake.

In this report a method is developed for obtaining product solutions to a certain class of partial differential equations. It is shown that, if the function which describes the distribution of depth of a lake or sea belongs to a certain quite general class of functions, then the technique developed herein can be used to reduce the problem of solving the partial differential equation and boundary conditions which describe the wind-driven circulation in the lake or sea to the problem of solving an ordinary differential

equation and its boundary conditions. To illustrate the principles involved, the technique is used to obtain the complete analytical solution for an elliptically shaped body of water with a particular choice of bottom topography when the depth is greater than half the Ekman friction depth.

The report begins by considering the general canonical form of the second-order linear elliptic partial differential equation in two independent variables. It is shown that if the two combinations of coefficients of the equation which are known as the invariants have a certain functional form then the equation can be transformed into an equation whose coefficients are functions of only one of its independent variables. Equations of this type, although they are not necessarily separable, possess infinitely many solutions which are products of a function of one of the variables with a function of the other. These solutions can be found by solving a single ordinary differential equation. Thus, an infinite family of solutions to the original differential equation can be found. It is anticipated that this family is sufficiently rich to be complete. However, instead of pursuing these matters further in this report, a new transformation of the independent variables is introduced. The transformed equation is again one whose coefficients depend on only one of the independent variables. A quite general boundary value problem for this equation is then posed in the interior of the unit circle in the plane for which the new variables are polar coordinates. It is then shown that by taking finite Fourier transforms the problem can be reduced to the problem of solving a second-order ordinary differential equation.

Next, the partial differential equation which describes the wind-driven circulation in a shallow sea or lake is introduced. A simplified form of this equation which is valid when the depth of the body of water is larger than half the Ekman friction depth is also given. It is shown that whenever the bottom topography of the body of water can be described by a function which belongs to a certain general class of functions these equations are of the type discussed herein. It is also shown that if, in addition, the depth of the lake is constant along the shore then the boundary value problem for determining the wind-driven circulation in the body of water can be transformed into a boundary value problem of the type discussed above and can, therefore, be reduced to the problem of solving an ordinary differential equation. The coefficients in this equation depend on the bottom topography. The ordinary differential equation together with its boundary conditions are solved for a particular choice of the bottom topography. Finally, the complete solution to the problem is obtained for an elliptically shaped lake or sea. The transformations which are carried out to obtain the solutions are best accomplished by using certain techniques of conformal mapping.

## SYMBOLS

$A$	function of $h/d$ given by eq. (42)
$a$	function of $x, y$
$\tilde{a}$	defined in eq. (4)
$B$	function of $h/d$ given by eq. (42)
$b$	function of $x, y$
$\tilde{b}$	defined in eq. (4)
$C$	function of $h/d$ defined in eq. (37)
$\mathcal{C}$	boundary of sea or lake
$c$	function of $x, y$
$\tilde{c}$	defined in eq. (4)
$D$	function of $h/d$ defined in eq. (37)
$d$	Ekman friction depth, $\pi(2\nu/f_c)^{1/2}$
$E, F$	functions of $h/d$ defined in eq. (38)
$\mathcal{F}$	defined by eq. (17)
$f$	function of $x, y$
$f_c$	Coriolis parameter
$g$	acceleration due to gravity
$H$	defined in eq. (49)
$H_s$	value of $H$ at lake or sea boundary (a constant)
$h$	depth of lake or sea
$I$	defined in eq. (114)
$\mathcal{I}$	invariant of linear, elliptic partial differential equation, defined in eq. (5)
$J_n$	Bessel function of first kind of order $n$
$\mathcal{J}$	invariant of linear, elliptic partial differential equation, defined in eq. (6)
$K$	complete elliptic integral of first kind
$K'$	$K(k')$
$\mathcal{K}_n$	kernel in solution for $\Omega_n(r)$ , eq. (104)
$\mathcal{K}_0$	kernel in solution for $\Omega_0(r)$ , eq. (94)

$k$	modulus of elliptic integral
$k'$	$\sqrt{1 - k^2}$
$L_1, L_2$	semimajor and semiminor axes of ellipse
$l_0, l_1$	constants of integration
$l_1^{(n)}, l_2^{(n)}$	constants, eq. (96)
$M_n$	constant
$n$	integer
$\hat{n}$	unit normal to sea or lake boundary
$n_1, n_2$	x- and y-components of $\hat{n}$
$p, q$	constants
$Q_1, Q_2$	x- and y-components of volume flow
$q_1, q_2$	x- and y-components of velocity
$r$	radial distance in $\chi$ -plane
$r_n$	$\lambda_n r$
$\tilde{r}$	dummy variable
$S$	length of curve in $z$ -plane
$s$	constant
$t$	complex variable, $\xi + i\eta$
$U$	dependent variable
$u$	harmonic function of $x, y$
$V$	dependent variable
$v$	harmonic conjugate to $u$
$W$	Wronskian
$w$	complex function, $u + iv$
$x$	independent variable, coordinate of lake-surface plane
$Y_n$	Bessel function of second kind of order $n$
$y$	independent variable, coordinate of lake-surface plane
$Z$	vertical coordinate
$z$	$x + iy$

$z_0$	a particular point interior to $\mathcal{C}$
$\alpha, \beta$	defined in eq. (51)
$\Gamma_n$	Fourier coefficient of $\mathcal{F}(r, \theta)$ , eq. (78)
$\gamma$	periodic function of $\theta$
$\gamma_n$	Fourier coefficient of $\gamma(\theta)$
$\delta$	parameter in depth distribution, eq. (84)
$\xi$	displacement of lake or sea surface
$\eta$	imaginary part of $t$
$\theta$	angular coordinate in $\chi$ -plane
$\vartheta$	wind direction
$\lambda$	complex constant
$\lambda_n$	defined in eq. (95)
$\mu_n$	defined in eq. (128)
$\nu$	coefficient of vertical eddy viscosity
$\xi$	real part of $t$
$\rho$	radial polar coordinate
$\sigma$	$\sigma^{(1)} + i\sigma^{(2)}$
$\sigma^{(1)}, \sigma^{(2)}$	linear combinations of wind stress, eq. (36)
$\tau_1, \tau_2$	x- and y-components of wind stress
$\tau_0^\dagger$	magnitude of dimensionless wind stress
$\tau_1^\dagger, \tau_2^\dagger$	$2\pi\tau_1/gd, 2\pi\tau_2/gd$
$\varphi$	differentiable function of $u$
$\chi$	defined in eq. (20)
$\psi$	function of $u$
$\Omega_n$	Fourier coefficient of $V$ , eq. (27)
$\Omega_n^h$	complementary function for $\Omega_n$
$\Omega_n^p$	particular solution of differential equation for $\Omega_n$
$\omega$	function of $x, y$
$\omega_n^{(1)}, \omega_n^{(2)}$	defined by eq. (102)



Subscripts:

$\left. \begin{array}{l} r, \theta \\ u, v \\ x, y \end{array} \right\}$  partial derivatives

Superscript:

\* denotes complex conjugate

## SOLUTIONS OF A CERTAIN GENERAL CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

The most general, second order, linear, elliptic, partial differential equation in two independent variables can always be reduced to the canonical form

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + a \frac{\partial U}{\partial x} + b \frac{\partial U}{\partial y} + cU = f \quad (1)$$

where the coefficients  $a$ ,  $b$ , and  $c$  and the nonhomogeneous term  $f$  are functions of  $x$  and  $y$ .

Any change of the dependent variable of the form

$$V = e^{\omega} U \quad (2)$$

where  $\omega$  is any sufficiently differentiable function of  $x$  and  $y$  transforms equation (1) into an equation of the same type. Thus, the function  $V$  satisfies the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \tilde{a} \frac{\partial V}{\partial x} + \tilde{b} \frac{\partial V}{\partial y} + \tilde{c}V = e^{\omega} f \quad (3)$$

where the coefficients  $\tilde{a}$ ,  $\tilde{b}$ , and  $\tilde{c}$  are related to the coefficients  $a$ ,  $b$ , and  $c$  of the original equation (1) by

$$\left. \begin{aligned} \tilde{a} &= a - 2\omega_x \\ \tilde{b} &= b - 2\omega_y \\ \tilde{c} &= c - (\omega_{xx} + \omega_{yy}) - (a\omega_x + b\omega_y) + (\omega_x)^2 + (\omega_y)^2 \end{aligned} \right\} \quad (4)$$

It is easy to show (ref. 2) by eliminating the function  $\omega$  between these relations that

$$\begin{aligned} \tilde{a}_y - \tilde{b}_x &= a_y - b_x \\ \tilde{c} - \frac{1}{2}(\tilde{a}_x + \tilde{b}_y) - \frac{1}{4}(\tilde{a}^2 + \tilde{b}^2) &= c - \frac{1}{2}(a_x + b_y) - \frac{1}{4}(a^2 + b^2) \end{aligned}$$

Thus, the quantities  $\mathcal{I}$  and  $\mathcal{J}$  defined in terms of the coefficients of any equation of the form equation (1) by

$$\mathcal{I} = a_y - b_x \quad (5)$$

$$\mathcal{J} = c - \frac{1}{2}(a_x + b_y) - \frac{1}{4}(a^2 + b^2) \quad (6)$$

are invariant under a transformation of the type shown in equation (2). They are, therefore, called the invariants of the differential equation (1) (ref. 3, vol. VI, art. 191).

We shall now restrict the discussion to the class of all equations of the type in equation (1) whose invariants  $\mathcal{I}$  and  $\mathcal{J}$  can be expressed in the form

$$\mathcal{I} = \varphi'(u) \left[ (u_x)^2 + (u_y)^2 \right] \quad (7)$$

$$\mathcal{J} = \psi(u) \left[ (u_x)^2 + (u_y)^2 \right] \quad (8)$$

where  $u$  is any nonconstant harmonic function of  $x$  and  $y$  (i. e.,  $u_{xx} + u_{yy} = 0$ ),  $\psi$  is any function of  $u$  and  $\varphi$  is any differentiable function of  $u$ . Since  $u$  satisfies Laplace's equation, it is easy to see from definition (5) and equation (7) that

$$\frac{\partial}{\partial y} \frac{1}{2} \left[ a - u_y \varphi(u) \right] = \frac{\partial}{\partial x} \frac{1}{2} \left[ b + u_x \varphi(u) \right]$$

This shows that there exists a differentiable function  $\omega$  of  $x$  and  $y$  which is deter-

mined to within an unimportant constant by

$$\left. \begin{aligned} \omega_x &= \frac{1}{2} [a - u_y \varphi(u)] \\ \omega_y &= \frac{1}{2} [b + u_x \varphi(u)] \end{aligned} \right\} \quad (9)$$

Hence, when the change of variable (2), with  $\omega$  determined from equations (9), is applied to equation (1) it follows that the coefficients  $\tilde{a}$ ,  $\tilde{b}$ , and  $\tilde{c}$  of the resulting equation (3) are given by

$$\begin{aligned} \tilde{a} &= u_y \varphi(u) \\ \tilde{b} &= -u_x \varphi(u) \\ \tilde{c} &= \left[ \psi(u) + \frac{1}{4} \varphi^2(u) \right] \left[ (u_x)^2 + (u_y)^2 \right] \end{aligned}$$

Hence,  $V$  satisfies the equation

$$V_{xx} + V_{yy} + \varphi(u)(u_y V_x - u_x V_y) + \left[ \psi(u) + \frac{1}{4} \varphi^2(u) \right] \left[ (u_x)^2 + (u_y)^2 \right] V = e^{\omega} f \quad (10)$$

Now let  $v$  be the harmonic conjugate<sup>1</sup> to  $u$ . Then  $u$  and  $v$  satisfy the Cauchy-Riemann equations

$$\left. \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \right\} \quad (11)$$

It follows from these equations that the Jacobian  $\partial(u, v)/\partial(x, y)$  is given by

$$\frac{\partial(u, v)}{\partial(x, y)} \equiv \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = (u_x)^2 + (u_y)^2 \quad (12)$$

---

<sup>1</sup>Thus  $v$  satisfies Laplace's equation and the function  $w = u + iv$  is an analytic function of the complex variable  $z = x + iy$ .

Since by hypothesis  $u$  is nonconstant, it follows from equation (12) that

$$\frac{\partial(u, v)}{\partial(x, y)} \neq 0$$

Hence, equation (10) can be further transformed by changing the independent variables  $x$  and  $y$  to the independent variables  $u$  and  $v$ . Now it is well known (ref. 4, p. 187) that under this change of variable the Laplacean of  $V$  transforms as<sup>2</sup>

$$V_{xx} + V_{yy} = (V_{uu} + V_{vv}) [(u_x)^2 + (u_y)^2] \quad (13)$$

Since the chain rule implies that

$$V_x = V_u u_x + V_v v_x$$

$$V_y = V_u u_y + V_v v_y$$

it follows from the Cauchy-Riemann equations (11) that

$$u_y V_x - u_x V_y = -V_v [(u_x)^2 + (u_y)^2] \quad (14)$$

It is easy to show that<sup>3</sup>

---

<sup>2</sup>The convention of writing  $V(u, v)$  in place of  $V(x(u, v), y(u, v))$  will be adopted since no confusion is likely to result from this.

<sup>3</sup>Equations (11) imply that  $x$  and  $y$  also satisfy the Cauchy-Riemann equations with respect to  $u$  and  $v$ . Thus,  $x_u = y_v$  and  $y_u = -x_v$ . Hence,

$$\frac{\partial(x, y)}{\partial(u, v)} = (x_u)^2 + (x_v)^2$$

But

$$\frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)} = 1$$

Hence,

$$(u_x)^2 + (u_y)^2 = \frac{1}{(x_u)^2 + (x_v)^2}$$

$$(x_u)^2 + (x_v)^2 = \frac{1}{(u_x)^2 + (u_y)^2} \quad (15)$$

Upon substituting equations (13) to (15) into equation (10) we find, since  $(u_x)^2 + (u_y)^2 \neq 0$ , that  $V$  must satisfy the equation

$$V_{uu} + V_{vv} - \varphi(u) V_v + \left[ \psi(u) + \frac{1}{4} \varphi^2(u) \right] V = \mathcal{F} \quad (16a)$$

where

$$\mathcal{F} \equiv \left[ (x_u)^2 + (x_v)^2 \right] e^{\omega_f} \quad (16b)$$

Notice that the coefficients of this equation are functions only of the independent variable  $u$  and not of the independent variable  $v$ . Then direct substitution shows that, for every real or complex constant  $\lambda$ , the associated homogeneous equation (i. e., eq. (16a) with  $\mathcal{F} = 0$ ) possesses a solution of the form

$$V(u, v) = e^{\lambda v} U_\lambda(u) \quad (17)$$

provided that  $U_\lambda$  satisfies the ordinary differential equation

$$U_\lambda'' + \left[ \left( \lambda - \frac{1}{2} \varphi \right)^2 + \psi \right] U_\lambda = 0$$

Of course, any linear combination of these "elementary" solutions is also a solution. However, the question of whether these elementary solutions form a complete set of solutions (i. e., whether every solution to eq. (16a) with  $\mathcal{F} = 0$  can be expressed as a linear combination of these elementary solutions) is beyond the scope of this report. The use of linear combinations of elementary solutions of the type shown in equation (17) to obtain solutions to specific boundary value problems involving homogeneous equations whose coefficients depend only on one of the independent variables is illustrated in references 5 and 6. Instead of pursuing these matters further, we shall turn to a discussion of an important special type of boundary value problem for the nonhomogeneous equation (16a). Before this is done, however, it will be convenient to introduce one further change of variable.

## SOLUTIONS OF A CERTAIN TYPE OF BOUNDARY VALUE PROBLEM

To this end notice that, since  $u$  and  $v$  are conjugate harmonic functions, the function

$$w = u + iv \quad (18)$$

is an analytic function of the complex variable

$$z = x + iy \quad (19)$$

Hence, the function  $\chi$  defined by

$$\chi = e^w \quad (20)$$

is also an analytic function of the complex variable  $z$ . Expressing  $\chi$  in the polar form

$$\chi = re^{i\theta} \quad (21)$$

reveals that

$$\left. \begin{aligned} u &= \ln r \\ v &= \theta \end{aligned} \right\} \quad (22)$$

When the change of variables defined by equation (22) is introduced into equation (16a), we obtain (where the convention introduced previously of writing  $\varphi(r)$  in place of  $\varphi(\ln r)$  etc. is used)

$$r \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial \theta^2} - \varphi(r) \frac{\partial V}{\partial \theta} + \left[ \psi(r) + \frac{1}{4} \varphi^2(r) \right] V = \mathcal{F}(r, \theta) \quad (23)$$

We shall now consider the following boundary value problem for equation (23) in the unit circle of the  $\chi$ -plane shown in figure 1.

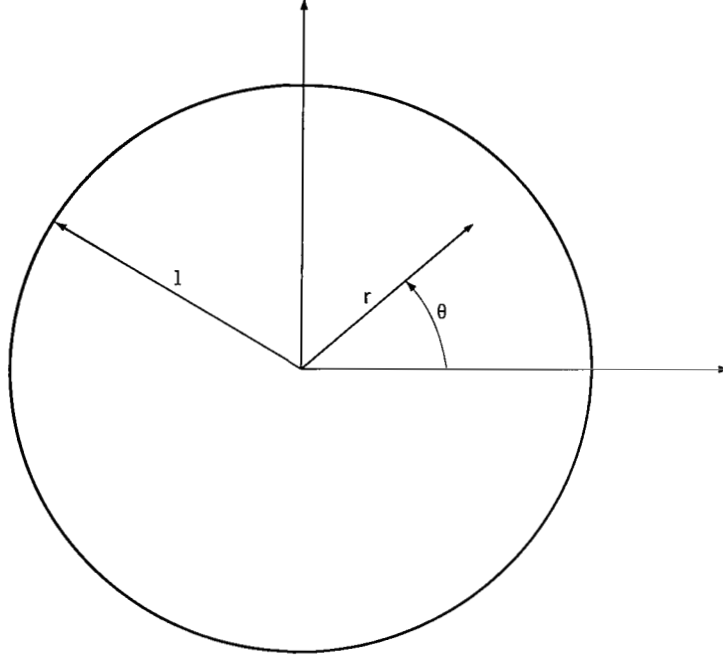


Figure 1. -  $z$ -plane.

$$r(rV_r)_r + V_{\theta\theta} - \varphi(r)V_\theta + \left[ \psi(r) + \frac{1}{4}\varphi^2(r) \right] V = \mathcal{F} \begin{cases} 0 \leq r \leq 1 \\ -\pi \leq \theta \leq \pi \end{cases} \quad (24)$$

$$pV_r + qV_\theta + sV = \gamma(\theta) \quad \text{for } r = 1, -\pi \leq \theta \leq \pi \quad (25)$$

where  $p$ ,  $q$ , and  $s$  are constants and  $\gamma$  is a periodic function of  $\theta$ . We require that  $V$  be a continuously differentiable function within the unit circle. Thus, in particular  $V(0, \theta)$  must be finite and independent of  $\theta$ , and the following periodicity conditions must hold:

$$\left. \begin{aligned} V(r, -\pi) &= V(r, \pi) \\ V_\theta(r, -\pi) &= V_\theta(r, \pi) \end{aligned} \right\} 0 \leq r \leq 1 \quad (26)$$

In order to obtain a solution to this problem put

$$\Omega_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} V(r, \theta) e^{-in\theta} d\theta \quad n = 0, \pm 1, \pm 2, \dots \quad (27)$$

Then it follows from the theory of Fourier series that this transform can be inverted to obtain

$$V(r, \theta) = \sum_{n=-\infty}^{\infty} \Omega_n(r) e^{in\theta} \quad (28)$$

Upon integrating by parts, we find that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} V_{\theta}(r, \theta) e^{-in\theta} d\theta = \frac{(-1)^n}{2\pi} [V(r, \pi) - V(r, -\pi)] + in\Omega_n(r)$$

Hence, the first periodicity condition in equation (26) shows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} V_{\theta}(r, \theta) e^{-in\theta} d\theta = in\Omega_n(r) \quad (29)$$

In a similar way it follows from the second periodicity condition in equation (26) that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} V_{\theta\theta}(r, \theta) e^{-in\theta} d\theta = -n^2 \Omega_n(r) \quad (30)$$

We shall now show that each function  $\Omega_n$  can be determined as the solution of a certain ordinary differential equation. To this end, multiply equations (24) and (25) by  $e^{-in\theta}/2\pi$  and integrate both sides of the resulting expressions with respect to  $\theta$  between  $-\pi$  and  $\pi$ . We find, upon making the definitions

$$\left. \begin{aligned} \Gamma_n(r) &\equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{F}(r, \theta) e^{-in\theta} d\theta \\ \gamma_n &\equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma(\theta) e^{-in\theta} d\theta \end{aligned} \right\} \quad n = 0, \pm 1, \pm 2, \dots \quad (31)$$

and using equations (29) and (30), that  $\Omega_n$  must satisfy the ordinary differential equation



$$r(r\Omega'_n)' + \left(\psi + \frac{1}{4}\varphi^2 - n^2 - \ln\varphi\right)\Omega_n = \Gamma_n \quad n = 0, \pm 1, \pm 2, \dots \quad (32)$$

subject to the boundary condition

$$p\Omega'_n(1) + (s + \ln q)\Omega_n(1) = \gamma_n \quad n = 0, \pm 1, \pm 2, \dots \quad (33)$$

Equation (32) and the single boundary condition in equation (33) are not sufficient to completely determine  $\Omega_n$ . Notice, however, that the origin ( $r = 0$ ) is a regular singular point of the ordinary differential equation (32) (we are, of course, assuming that the functions  $\varphi$  and  $\psi$  are bounded at  $r = 0$ ). Hence, one of the homogeneous solutions to this equation will be bounded at  $r = 0$  while the other will be unbounded (ref. 7, pp. 101-108). But since  $V$  must be bounded at  $r = 0$ , each  $\Omega_n$  must also be bounded. Hence, there must exist (for each integer  $n$ ) a finite number  $M_n$  such that

$$\lim_{r \rightarrow 0} \Omega_n(r) < M_n \quad n = 0, \pm 1, \pm 2, \dots \quad (34)$$

and the solution to the nonhomogeneous equation (32) which satisfies the boundedness condition in equation (34) will involve only one arbitrary constant. This constant is determined by the boundary condition shown in equation (33). Thus, the Fourier coefficients  $\Omega_n$  are completely determined by the differential equations (32) together with the conditions in equations (33) and (34). When these solutions  $\Omega_n$  for  $n = 0, \pm 1, \pm 2, \dots$  have been found, they can be substituted into equation (28) to yield the complete solution to the boundary value problem posed by equations (24) and (25).

## BASIC EQUATIONS FOR WIND-DRIVEN CIRCULATION IN SHALLOW SEA OR LAKE

The basic equation governing the wind-driven circulation in a one-layer shallow sea or lake was derived by Welander (ref. 1). The basic assumptions used in deriving the equation from the Navier-Stokes equations are that the water density is constant, the vertical eddy viscosity is independent of depth but dependent on wind velocity, the pressure is hydrostatic, and the lateral friction and nonlinear acceleration terms can be neglected. A detailed discussion of the derivation is contained in reference 8. Welander wrote the equation with the surface displacement as the dependent variable. This equation for the surface displacement  $\zeta$  of a lake or sea of variable depth  $h$  is (ref. 1)

$$\begin{aligned}
(ghE)(\zeta_{xx} + \zeta_{yy}) + \left[ (ghE)_x + (ghF)_y \right] \zeta_x + \left[ (ghE)_y - (ghF)_x \right] \zeta_y \\
= - \left[ \frac{gd}{2\pi} \sigma^{(1)} \right]_x - \left[ \frac{gd}{2\pi} \sigma^{(2)} \right]_y
\end{aligned} \quad (35)$$

where  $x$  and  $y$  are the horizontal, locally Cartesian coordinates,  $g$  is the acceleration due to gravity,  $d = \pi \sqrt{2\nu/f_c}$  is the Ekman friction depth,  $\nu$  is the coefficient of vertical eddy viscosity, and  $f_c$  is the Coriolis parameter which introduces the effect of the Earth's rotation. In addition,

$$\left. \begin{aligned} \sigma^{(1)} &\equiv \frac{2\pi\tau_1}{gd} C - \frac{2\pi\tau_2}{gd} D \\ \sigma^{(2)} &\equiv \frac{2\pi\tau_1}{gd} D + \frac{2\pi\tau_2}{gd} C \end{aligned} \right\} \quad (36)$$

where  $\tau_1$  and  $\tau_2$  are, respectively, the  $x$ - and  $y$ -components of the wind stress and

$$\left. \begin{aligned} C &\equiv \frac{2 \sin\left(\frac{\pi h}{d}\right) \sinh\left(\frac{\pi h}{d}\right)}{\cos\left(\frac{2\pi h}{d}\right) + \cosh\left(2\pi \frac{h}{d}\right)} \\ D &\equiv \frac{2 \cos\left(\pi \frac{h}{d}\right) \cosh\left(\pi \frac{h}{d}\right)}{\cos\left(2\pi \frac{h}{d}\right) + \cosh\left(2\pi \frac{h}{d}\right)} - 1 \end{aligned} \right\} \quad (37)$$

$$\left. \begin{aligned} E &\equiv \frac{1}{\left(2\pi \frac{h}{d}\right)} \left[ \frac{\sin\left(2\pi \frac{h}{d}\right) - \sinh\left(2\pi \frac{h}{d}\right)}{\cos\left(2\pi \frac{h}{d}\right) + \cosh\left(2\pi \frac{h}{d}\right)} \right] \\ F &\equiv - \frac{1}{\left(2\pi \frac{h}{d}\right)} \left[ \frac{\sin\left(2\pi \frac{h}{d}\right) + \sinh\left(2\pi \frac{h}{d}\right)}{\cos\left(2\pi \frac{h}{d}\right) + \cosh\left(2\pi \frac{h}{d}\right)} \right] + 1 \end{aligned} \right\} \quad (38)$$

The x- and y-components of the total volume flow  $Q_1$  and  $Q_2$ , respectively, are given in terms of the surface displacement in reference 1 as

$$\left. \begin{aligned} Q_1 &\equiv \int_{-h}^0 q_1 dZ \equiv \frac{gd}{2\pi f_c} \left[ \sigma^{(1)} + \frac{2\pi h}{d} (E\zeta_x - F\zeta_y) \right] \\ Q_2 &\equiv \int_{-h}^0 q_2 dZ \equiv \frac{gd}{2\pi f_c} \left[ \sigma^{(2)} + \frac{2\pi h}{d} (F\zeta_x + E\zeta_y) \right] \end{aligned} \right\} \quad (39)$$

The boundary condition for equation (35) is obtained by requiring that the normal volume flow vanish at the coasts. Thus, for any closed lake or sea let  $n_1$  and  $n_2$  denote the x and y components, respectively, of the outward drawn unit normal  $\hat{n}$  to its boundary. Then the boundary condition for equation (35) is

$$Q_1 n_1 + Q_2 n_2 = 0 \quad (40)$$

at all points of the boundary of the body of water.

It is convenient to rearrange equation (35) to obtain

$$\zeta_{xx} + \zeta_{yy} + \left[ A \left( 2\pi \frac{h}{d} \right)_x - B \left( 2\pi \frac{h}{d} \right)_y \right] \zeta_x + \left[ A \left( 2\pi \frac{h}{d} \right)_y + B \left( 2\pi \frac{h}{d} \right)_x \right] \zeta_y = f \quad (41)$$

where

$$\left. \begin{aligned} A &\equiv \frac{2 \sin \left( 2\pi \frac{h}{d} \right) \sinh \left( 2\pi \frac{h}{d} \right)}{\left[ \cosh \left( 2\pi \frac{h}{d} \right) + \cos \left( 2\pi \frac{h}{d} \right) \right] \left[ \sinh \left( 2\pi \frac{h}{d} \right) - \sin \left( 2\pi \frac{h}{d} \right) \right]} \\ B &\equiv \frac{\sinh \left( 2\pi \frac{h}{d} \right) + \sin \left( 2\pi \frac{h}{d} \right)}{\cosh \left( 2\pi \frac{h}{d} \right) + \cos \left( 2\pi \frac{h}{d} \right)} \end{aligned} \right\} \quad (42)$$

$$f \equiv -\frac{1}{2\pi \frac{h}{d} E} \left[ \sigma_x^{(1)} + \sigma_y^{(2)} \right] \quad (43)$$

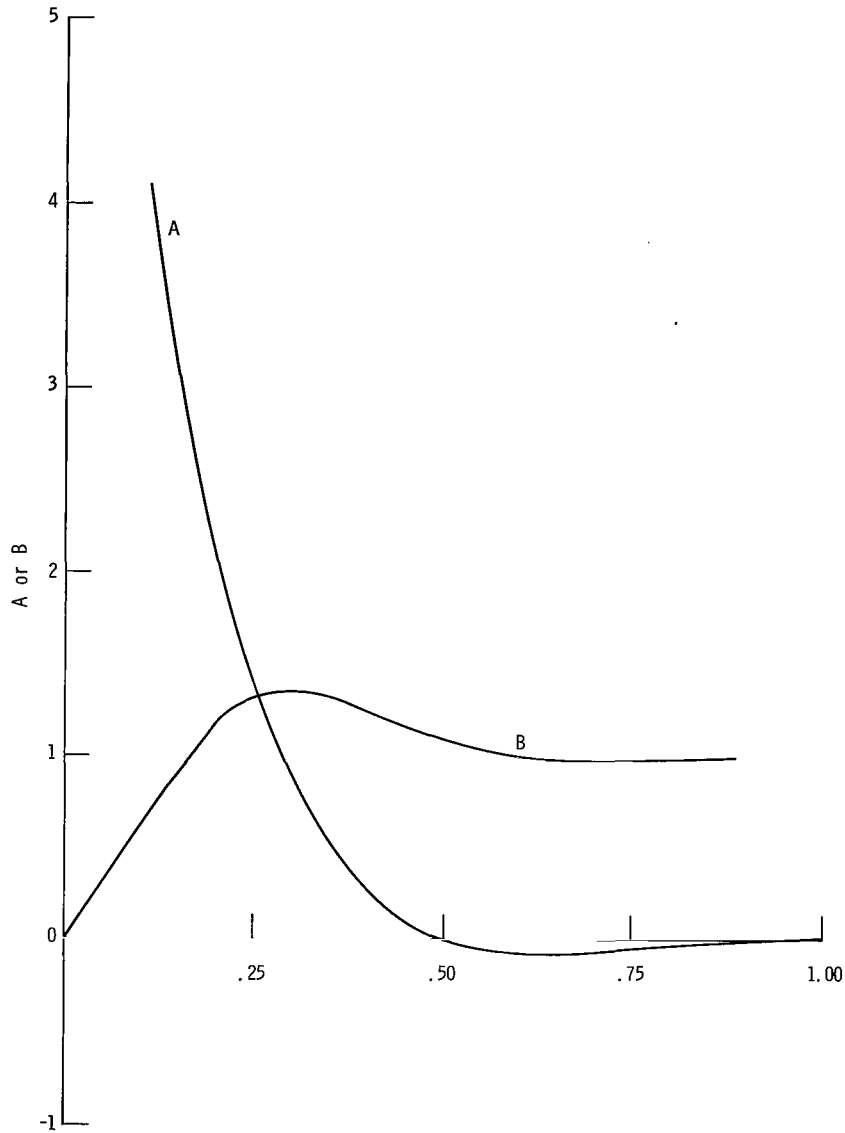


Figure 2. - Coefficients A and B.

The coefficients A and B are plotted as a function of  $h/d$  in figure 2. It is easy to see from this figure that for  $h/d > 1/2$  to a fairly close approximation

and

$$\left. \begin{array}{l} A \approx 0 \\ B \approx 1 \end{array} \right\} \text{ for } \frac{h}{d} > \frac{1}{2} \quad (44)$$

The quantities  $2\pi(h/d)(1 - F)$  and  $2\pi(h/d)E$  are shown as functions of  $h/d$  in figure 3.

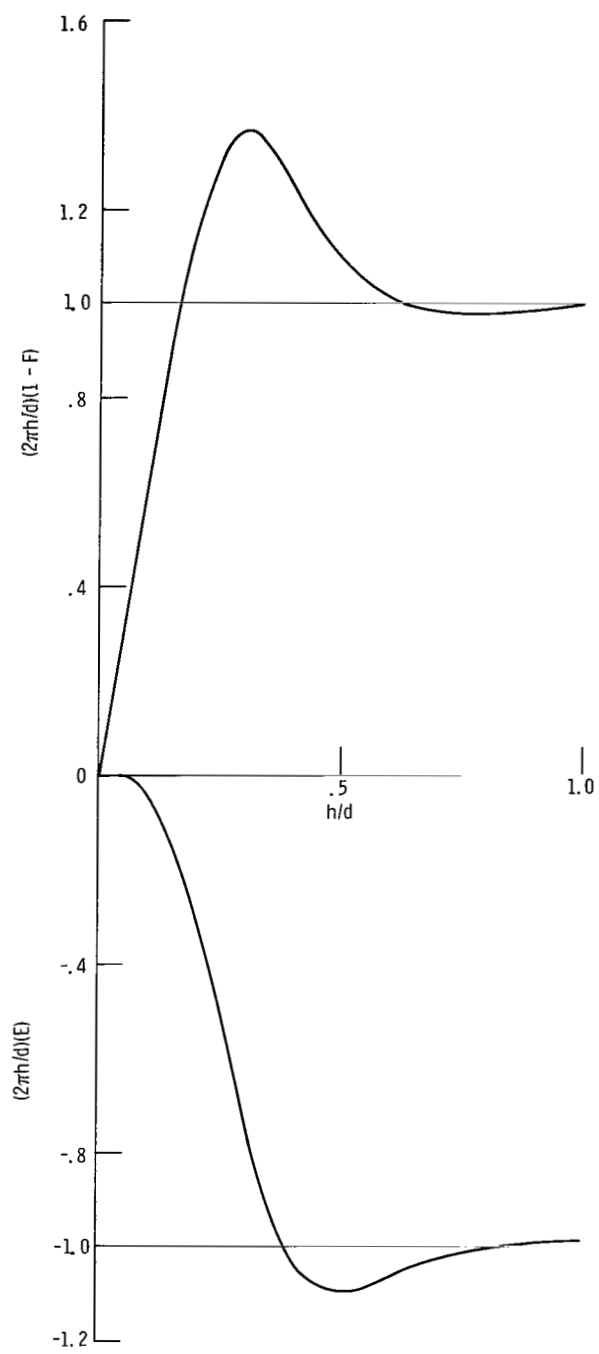


Figure 3. - Coefficients  $(2\pi h/d)(E)$  and  $(2\pi h/d)(1 - F)$ .

It can be seen from this figure that for  $h/d > 1/2$

$$2\pi \frac{h}{d} F \approx 2\pi \frac{h}{d} - 1$$

$$2\pi \frac{h}{d} E \approx -1$$

Thus, for  $h/d > 1/2$  expression (43) for  $f$  and equation (39) for the components of the total volume flow can be approximated fairly closely by the simpler expressions

$$f \approx \sigma_x^{(1)} + \sigma_y^{(2)} \quad \text{for } \frac{h}{d} > \frac{1}{2} \quad (45)$$

and

$$\left. \begin{aligned} Q_1 &\approx \frac{gd}{2\pi f_c} \left[ \sigma^{(1)} - \xi_x + \left(1 - 2\pi \frac{h}{d}\right) \xi_y \right] \\ Q_2 &\approx \frac{gd}{2\pi f_c} \left[ \sigma^{(2)} - \xi_y - \left(1 - 2\pi \frac{h}{d}\right) \xi_x \right] \end{aligned} \right\} \quad \text{for } \frac{h}{d} > \frac{1}{2} \quad (46)$$

The coefficients  $C$  and  $D$  of equation (37) are plotted in figure 4. It is also shown in this figure that for  $h/d > 1/2$  the asymptotic forms

$$C \approx 2e^{-\pi h/d} \sin \frac{\pi h}{d}$$

$$D \approx 2e^{-\pi h/d} \cos \frac{\pi h}{d} - 1$$

agree with the exact expressions for  $C$  and  $D$  to a fairly close approximation. Hence, equation (36) for  $\sigma^{(1)}$  and  $\sigma^{(2)}$  can be approximated by

$$\left. \begin{aligned} \sigma^{(1)} &\approx \frac{2\pi\tau_1}{gd} 2e^{-\pi h/d} \sin \frac{\pi h}{d} + \frac{2\pi\tau_2}{gd} \left(1 - 2e^{-\pi h/d} \cos \frac{\pi h}{d}\right) \\ \sigma^{(2)} &\approx \frac{2\pi\tau_1}{gd} \left(2e^{-\pi h/d} \cos \frac{\pi h}{d} - 1\right) + \frac{2\pi\tau_2}{gd} 2e^{-\pi h/d} \sin \frac{\pi h}{d} \end{aligned} \right\} \quad \text{for } \frac{h}{d} > \frac{1}{2} \quad (47)$$

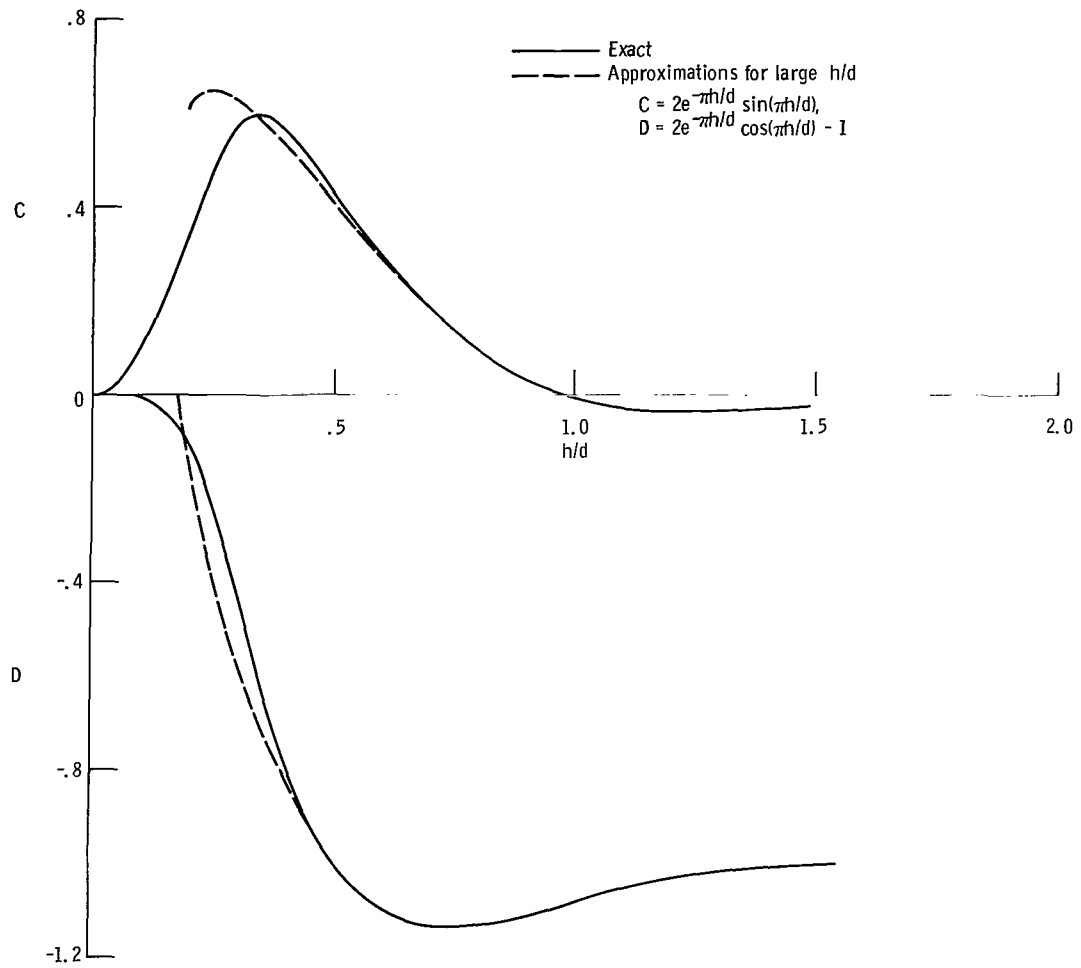


Figure 4. - Coefficients C and D.

It can also be seen from figure 4 that, for  $h/d > 1$ , C and D can be further approximated by the limits for  $h/d \rightarrow \infty$ , namely,

$$C \approx 0$$

and

$$D \approx -1$$

with a fair degree of accuracy. Hence, equation (36) can be replaced by the even simpler expressions

$$\left. \begin{aligned} \sigma^{(1)} &\approx \frac{2\pi\tau_2}{gd} \\ \sigma^{(2)} &\approx -\frac{2\pi\tau_1}{gd} \end{aligned} \right\} \text{ for } \frac{h}{d} > 1 \quad (48)$$

## CLASS OF DEPTH DISTRIBUTIONS FOR WHICH EQUATIONS CAN BE SOLVED BY FINITE FOURIER TRANSFORMS

We shall now suppose that there exists a nonconstant harmonic function  $u$  of  $x$  and  $y$  and an arbitrary function  $H$  of  $u$  only such that the depth distribution  $h$  can be expressed in the form

$$\frac{h(x, y)}{d} = \frac{1}{2\pi} H[u(x, y)] \quad (49)$$

This is a fairly general functional form and it will be possible, for any one of a large number of lakes and seas, to choose the functions  $u$  and  $H$  in equation (49) so that the depth distribution is approximated fairly closely by a relation of this type.

When the relation (49) is substituted into equations (41) and (42), we find that

$$\zeta_{xx} + \zeta_{yy} + 2 \left[ \alpha(u)u_x - \beta(u)u_y \right] \zeta_x + 2 \left[ \alpha(u)u_y + \beta(u)u_x \right] \zeta_y = f \quad (50)$$

where

$$\left. \begin{aligned} \alpha &\equiv \frac{\sin H \sinh H}{(\cosh H + \cos H)(\sinh H - \sin H)} \frac{dH}{du} \\ \beta &\equiv \frac{1}{2} \frac{\sinh H + \sin H}{\cosh H + \cos H} \frac{dH}{du} \end{aligned} \right\} \quad (51)$$

$$f \equiv \frac{\cosh H + \cos H}{\sinh H - \sin H} \left[ \frac{\partial \sigma^{(1)}}{\partial x} + \frac{\partial \sigma^{(2)}}{\partial y} \right] \quad (52)$$

and it follows from the preceding discussion and equations (44) that, when  $h/d > 1/2$ , the coefficients  $\alpha$  and  $\beta$  are given to a close approximation by



$$\left. \begin{array}{l} \alpha \approx 0 \\ \beta \approx \frac{1}{2} \frac{dH}{du} \end{array} \right\} \text{ for } \frac{h}{d} > \frac{1}{2} \quad (53)$$

Now the invariants of equation (50) satisfy conditions (7) and (8) for all differentiable functions  $\alpha$  and  $\beta$ . Hence, this equation is of the type discussed in the preceding section. In order to show this, notice that since

$$\frac{\partial}{\partial y} [\alpha(u)u_x] = \frac{\partial}{\partial x} [\alpha(u)u_y]$$

it follows from definitions (5) and (6) that the invariants  $\mathcal{I}$  and  $\mathcal{J}$  of equation (50) are

$$\begin{aligned} \mathcal{I} &= -2 \left\{ \frac{\partial}{\partial y} [\beta(u)u_y] + \frac{\partial}{\partial x} [\beta(u)u_x] \right\} \\ \mathcal{J} &= - \left\{ \frac{\partial}{\partial x} [\alpha(u)u_x] + \frac{\partial}{\partial y} [\alpha(u)u_y] \right\} - \left\{ [\alpha(u)]^2 + [\beta(u)]^2 \right\} [(u_x)^2 + (u_y)^2] \end{aligned}$$

Or, since  $u$  satisfies Laplace's equation,

$$\mathcal{I} = -2 \frac{d\beta}{du} [(u_x)^2 + (u_y)^2] \quad (54)$$

$$\mathcal{J} = - \left( \frac{d\alpha}{du} + \alpha^2 + \beta^2 \right) [(u_x)^2 + (u_y)^2] \quad (55)$$

Thus, the invariants  $\mathcal{I}$  and  $\mathcal{J}$  of equation (50) satisfy conditions (7) and (8) with

$$\varphi(u) = -2\beta(u) \quad (56)$$

$$\psi(u) = - \left\{ [\alpha(u)]' + [\alpha(u)]^2 + [\beta(u)]^2 \right\} \quad (57)$$

Equation (50) can, therefore, be transformed into an equation of the type (16a) and, hence, also into an equation of type (23). Thus, it follows from equations (1), (9), (50), and (56) that

$$\omega_x = \alpha u_x$$

$$\omega_y = \alpha u_y$$

Hence,

$$\omega = \int \alpha(u) du$$

Therefore, definition (51) shows that

$$\omega = \int \frac{\sin H \sinh H}{(\cosh H + \cos H)(\sinh H - \sin H)} dH$$

or

$$\omega = \frac{1}{2} \ln \left( \frac{\sinh H - \sin H}{\cosh H + \cos H} \right) = \ln \sqrt{-HE} \quad (58)$$

and equation (38) and figure 3 show that

$$\omega \approx 0 \quad \text{if } \frac{h}{d} > \frac{1}{2} \quad (59)$$

Therefore, in view of equations (1), (2), (38), (41), and (49)

$$V = \sqrt{\frac{\sinh H - \sin H}{\cos H + \cosh H}} \zeta \quad (60)$$

and

$$V \approx \zeta \quad \text{if } \frac{h}{d} > \frac{1}{2} \quad (61)$$

The introduction of equations (56) and (57) into equation (17a) yields

$$V_{uu} + V_{vv} + 2\beta(u)V_v - \left\{ [\alpha(u)]' + [\alpha(u)]^2 \right\} V = \mathcal{F} \quad (62)$$

where, from equations (16b), (52), and (58),

$$\mathcal{F} = \left[ (x_u)^2 + (x_v)^2 \right] \sqrt{\frac{\cosh H + \cos H}{\sinh H + \sin H}} \left[ \sigma_x^{(1)} + \sigma_y^{(2)} \right] \quad (63)$$

Upon using the polar coordinates  $r$  and  $\theta$ , equation (23) specializes to

$$r \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial \theta^2} + 2\beta(r) \frac{\partial V}{\partial \theta} - \left\{ r \frac{d\alpha(r)}{dr} + [\alpha(r)]^2 \right\} V = \mathcal{F} \quad (64)$$

When  $h/d > 1/2$ , it follows from equations (53), (61), and (22), and figure 3 that this equation becomes approximately

$$\left. \begin{aligned} r \frac{\partial}{\partial r} \left( r \frac{\partial \xi}{\partial r} \right) + \frac{\partial^2 \xi}{\partial \theta^2} + r \frac{dH}{dr} \frac{\partial \xi}{\partial \theta} &= \mathcal{F} \\ \mathcal{F} &= \left[ (x_u)^2 + (x_v)^2 \right] \left[ \sigma_x^{(1)} + \sigma_y^{(2)} \right] \end{aligned} \right\} \text{ for } \frac{h}{d} > \frac{1}{2} \quad (65)$$

We shall now consider a lake or shallow sea bounded by a closed curve such as illustrated in figure 5. We shall suppose that the bottom topography of the basin can be approximately described by a function of the form (49) and that the depth along the shore  $\mathcal{C}$  is constant. Hence, the harmonic function  $u$  is constant along  $\mathcal{C}$ . Now let  $v$  be

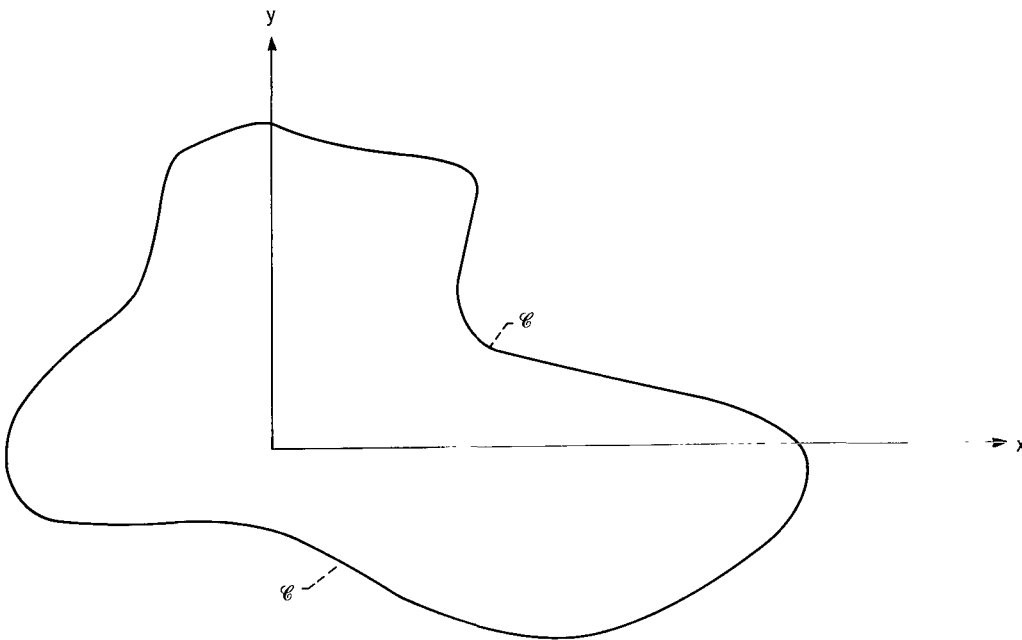


Figure 5. - Lake or sea configuration.

the harmonic conjugate to  $u$  and put

$$w = u + iv$$

Then  $w$  is an analytic function of  $z = x + iy$ , and the function  $\chi$  defined by equation (20) is also an analytic function of  $z$ . But

$$|\chi| = e^u$$

Hence,

$$|\chi(z)| = \text{constant} \quad \text{for } z \in \mathcal{C}$$

We can, without loss of generality, choose this constant to be unity. Since  $u$  and therefore  $\chi$  is nonconstant, the transformation

$$z \rightarrow \chi$$

maps the region occupied by the body of water onto either the interior or the exterior of the unit circle of the  $\chi$ -plane. By replacing  $u$  by  $-u$  and hence  $w$  by  $-w$ , if necessary, we can further suppose with no loss of generality that the mapping

$$z \rightarrow \chi$$

transforms the region occupied by the body of water onto the interior of the unit circle. Notice that  $\mathcal{C}$  is mapped onto the circle. The region occupied by the body of water in the  $\chi$ -plane is shown in figure 1. The existence of this function is a direct consequence of the Riemann mapping theorem which shows that every simply connected region bounded by a closed curve  $\mathcal{C}$  can be mapped conformally into the interior of the unit circle. In addition, this mapping can always be chosen so that any given point in the interior of  $\mathcal{C}$  maps into the origin of the unit circle and any direction through the point corresponds to the direction of the positive real axis. Once this is done the mapping is unique. In fact, if

$$z \rightarrow \chi_1$$

is any mapping of the interior of  $\mathcal{C}$  into the interior of the unit circle, and if  $z_0$  is any point in the interior of  $\mathcal{C}$ , then the mapping  $z \rightarrow \chi_2$  defined by

$$\chi_2(z) = \frac{\chi_1(z) - \chi_1(z_0)}{1 - \chi_1^*(z_0)\chi_1(z)}$$

also transforms the interior of  $\mathcal{C}$  into the interior of the unit circle but in this case the point  $z_0$  maps into the center of the circle.

We shall now show that once the boundary value problem for determining the surface displacement is transformed into the  $\chi$ -plane it can be reduced to solving an ordinary differential equation.

It has been shown that the region occupied by the body of water maps into the interior of the unit circle in the  $\chi$ -plane and that the shore line maps into the unit circle itself. We have also shown that with the change of dependent variable in equation (60) and with the new independent variables taken to be the polar coordinates ( $r$  and  $\theta$ ) in the  $\chi$ -plane, differential equation (50) for the surface displacement transforms into equation (64) or, when  $h/d > 1/2$ , into the approximate equation (65). In order to completely transform the boundary value problem for determining the surface displacement into the  $\chi$ -plane, it remains only to transform boundary condition (40) into a boundary condition on the unit circle in the  $\chi$ -plane.

To this end notice that, since  $u$  is constant on  $\mathcal{C}$ , the normal  $\hat{n}$  to  $\mathcal{C}$  is given by

$$\hat{n} = \frac{\nabla u}{|\nabla u|}$$

Hence, it follows from the Cauchy-Riemann conditions that the components of  $\hat{n}$  can be written as

$$\left. \begin{aligned} n_1 &= \frac{1}{\left| \frac{dw}{dz} \right|} \operatorname{Re} \left( \frac{dw}{dz} \right)^* = \frac{1}{\left| \frac{dw}{dz} \right|} \operatorname{Re} \left( \frac{dw}{dz} \right)^* \frac{dw}{dz} \frac{dz}{dw} = \left| \frac{dw}{dz} \right| \operatorname{Re} \frac{dz}{dw} \\ \text{and} \\ n_2 &= \frac{1}{\left| \frac{dw}{dz} \right|} \operatorname{Im} \left( \frac{dw}{dz} \right)^* = \left| \frac{dw}{dz} \right| \operatorname{Im} \frac{dz}{dw} \end{aligned} \right\} \quad (66)$$

Thus, it follows from equations (39) and (49) that the boundary condition (40) can be written as

$$\mathcal{R}e \left\{ \left[ \sigma + H(E + iF) \left( \frac{\partial \xi}{\partial x} + i \frac{\partial \xi}{\partial y} \right) \right] \left( \frac{dz}{dw} \right)^* \right\} = 0$$

where we have put

$$\sigma \equiv \sigma^{(1)} + i\sigma^{(2)} \quad (67)$$

Upon using the Cauchy-Riemann equations and the chain rule for partial derivatives this becomes

$$\mathcal{R}e \left[ \sigma \left( \frac{dz}{dw} \right)^* + H(E + iF) \left( \frac{\partial \xi}{\partial u} + i \frac{\partial \xi}{\partial v} \right) \right] = 0$$

or

$$HF \frac{\partial \xi}{\partial v} - HE \frac{\partial \xi}{\partial u} = \mathcal{R}e \left[ \sigma \left( \frac{dz}{dw} \right)^* \right]$$

When  $h/d > 1/2$ , figure 3 shows that this becomes approximately

$$(H - 1) \frac{\partial \xi}{\partial v} + \frac{\partial \xi}{\partial u} = \mathcal{R}e \left[ \sigma \left( \frac{dz}{dw} \right)^* \right] \quad \text{for } \frac{h}{d} > \frac{1}{2}$$

It now follows from equation (60) that  $V$  must satisfy the boundary condition

$$HF \frac{\partial V}{\partial v} - HE \frac{\partial V}{\partial u} + \frac{1}{2} \frac{dHE}{du} V = \sqrt{\frac{\sinh H - \sin H}{\cos H + \cosh H}} \mathcal{R}e \left[ \sigma \left( \frac{dz}{dw} \right)^* \right]$$

Finally, upon introducing the new independent variables  $r$  and  $\theta$  given by equation (22) and recalling that the boundary  $\mathcal{C}$  maps into the unit circle in the  $\chi$ -plane, the boundary condition for  $V$  on the unit circle becomes

$$pV_r + qV_\theta + sV = \gamma(\theta) \quad \text{for } r = 1, -\pi \leq \theta \leq \pi \quad (68)$$

where

$$\left. \begin{aligned} p &= \frac{\sinh H(1) - \sin H(1)}{\cosh H(1) + \cos H(1)} \\ q &= H(1) - \frac{\sinh H(1) + \sin H(1)}{\cosh H(1) + \cos H(1)} \\ s &= - \frac{\sin H(1) \sinh H(1)}{[\cos H(1) + \cosh H(1)]^2} \frac{dH}{dr} \Big|_{r=1} \end{aligned} \right\} \quad (69)$$

$$\gamma(\theta) = \sqrt{\frac{\sinh H(1) - \sin H(1)}{\cos H(1) + \cosh H(1)}} \Re e \left[ \sigma(1, \theta) \left( \frac{dz}{dw} \right) \Big|_{r=1}^* \right] \quad (70)$$

When  $h/d > 1/2$ , equation (68) can be replaced by the approximate boundary condition

$$p\zeta_r + q\zeta_\theta = \gamma(\theta) \quad \text{for } r = 1, -\pi \leq \theta \leq \pi, \frac{h}{d} > \frac{1}{2} \quad (71)$$

where now

$$\left. \begin{aligned} p &= 1 \\ q &= H(1) - 1 \end{aligned} \right\} \quad (72)$$

and

$$\gamma(\theta) = \Re e \left[ \sigma(1, \theta) \left( \frac{dz}{dw} \right) \Big|_{r=1}^* \right] \quad (73)$$

We therefore now show that the surface displacement for a sea or lake of arbitrary shape can be found by solving differential equation (64) in the interior of the unit circle subject to boundary condition (68). Or, if  $h/d > 1/2$ , the surface displacement is approximately obtained by solving equation (65) in the unit circle subject to boundary condition (71).

## SUMMARY OF EQUATIONS IN $\chi$ -PLANE

Upon comparing equations (64) and (68) (or when  $h/d > 1/2$  the approximate eqs. (65) and (71)) with equations (24) and (25), we see that this boundary value problem is a special case of the one solved in the preceding section. Hence, it follows from equations (28) and (60) that the surface displacement is given by

$$\zeta = \sqrt{\frac{\cosh H + \cos H}{\sinh H - \sin H}} \sum_{n=-\infty}^{\infty} \Omega_n(r) e^{in\theta} \quad (74)$$

or, when  $h/d > 1/2$ , approximately by

$$\zeta = \sum_{n=-\infty}^{\infty} \Omega_n(r) e^{in\theta} \quad \text{for } \frac{h}{d} > \frac{1}{2} \quad (75)$$

where (as can be seen from eqs. (22), (32) to (34), (56), and (57)) each function  $\Omega_n(r)$  is determined by solving the ordinary differential equation

$$r(r\Omega'_n)' - \left(r \frac{d\alpha}{dr} + \alpha^2 + n^2 - 2in\beta\right) \Omega_n = \Gamma_n \quad \left\{ \begin{array}{l} n = 0, \pm 1, \pm 2, \dots, \\ 0 \leq r \leq 1 \end{array} \right. \quad (76)$$

subject to the boundary conditions

$$\left. \begin{array}{l} p\Omega'_n(1) + (s + inq)\Omega_n(1) = \gamma_n \\ \lim_{r \rightarrow 0} \Omega_n(r) \text{ is finite} \end{array} \right\} \quad n = 0, \pm 1, \pm 2, \dots \quad (77)$$

The right sides of equations (76) and (77) are given by (see eqs. (31), (63), and (70))



$$\left. \begin{aligned} \Gamma_n &= \frac{1}{2\pi} \sqrt{\frac{\cosh H + \cos H}{\sinh H - \sin H}} \int_{-\pi}^{\pi} e^{-in\theta} \left[ \sigma_x^{(1)} + \sigma_y^{(2)} \right] \left| \frac{dz}{dw} \right|^2 d\theta \\ \gamma_n &= \frac{1}{2\pi} \sqrt{\frac{\sinh H(1) - \sin H(1)}{\cosh H(1) + \cos H(1)}} \int_{-\pi}^{\pi} e^{-in\theta} \operatorname{Re} \left[ \sigma(1, \theta) \left( \frac{dz}{dw} \right)_{r=1}^* \right] d\theta \end{aligned} \right\} n = 0, \pm 1, \pm 2, \dots \quad (78)$$

or, if  $h/d > 1/2$  (see eqs. (31), (65), and (73)), these can be replaced by the approximate relations

$$\left. \begin{aligned} \Gamma_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \left[ \sigma_x^{(1)} + \sigma_y^{(2)} \right] \left| \frac{dz}{dw} \right|^2 d\theta \\ \gamma_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \operatorname{Re} \left[ \sigma(1, \theta) \left( \frac{dz}{dw} \right)_{r=1}^* \right] d\theta \end{aligned} \right\} n = 0, \pm 1, \pm 2, \dots \quad (79)$$

The coefficients  $\alpha$  and  $\beta$  are given by (see eqs. (51) and (22))

$$\left. \begin{aligned} \alpha &= \frac{\sin H \sinh H}{(\cosh H + \cos H)(\sinh H - \sin H)} r \frac{dH}{dr} \\ \beta &= \frac{1}{2} \frac{\sinh H + \sin H}{\cosh H + \cos H} r \frac{dH}{dr} \end{aligned} \right\} \quad (80)$$

and when  $h/d > 1/2$  these can be replaced by the approximate relations (see eqs. (22) and (53))

$$\left. \begin{aligned} \alpha &\approx 0 \\ \beta &\approx \frac{1}{2} r \frac{dH}{dr} \end{aligned} \right\} \text{for } \frac{h}{d} > \frac{1}{2} \quad (81)$$

Finally, the constants  $p$ ,  $q$ , and  $s$  are given by (see eq. (69))

$$\left. \begin{aligned} p &= \frac{\sinh H(1) - \sin H(1)}{\cosh H(1) + \cos H(1)} \\ q &= H(1) - \frac{\sinh H(1) + \sin H(1)}{\cosh H(1) + \cos H(1)} \\ s &= - \frac{\sin H(1) \sinh H(1)}{[\cos H(1) + \cosh H(1)]^2} \frac{dH}{dr} \Big|_{r=1} \end{aligned} \right\} \quad (82)$$

or when  $h/d > 1/2$  approximately by (see eq. (72))

$$\left. \begin{aligned} p &= 1 \\ q &= H(1) - 1 \\ s &= 0 \end{aligned} \right\} \quad (83)$$

Thus, an exact solution for the surface displacement (and since all other physical quantities can be expressed in terms of this, an exact solution to the complete flow problem) can be obtained once the ordinary differential equation (76) with boundary condition (77) has been solved. This solution is given parametrically in terms of the variables  $r$  and  $\theta$ . These parametric variables are determined in terms of the physical variables  $x$  and  $y$  by the mapping

$$z \rightarrow \chi$$

with  $z = x + iy$  and  $\chi = re^{i\theta}$ , which maps the lake or sea conformally into the unit circle with a selected point being mapped into the origin. The coefficients of the ordinary differential equation depend upon the particular choice of the function  $H(r)$ . Thus, it is impossible to proceed further in the general case.

The procedures involved in obtaining a complete solution are best illustrated by considering a particular case. In order to simplify the calculations it will be assumed that  $h/d > 1/2$  and approximate forms in equation (81) will be used for the coefficients  $\alpha$  and  $\beta$ . A particular depth distribution function  $H$  of  $r$  will first be chosen. The ordinary differential equation will then be solved and the surface displacement will be found as a function of  $r$  and  $\theta$ . A particular shape for the body of water will then be chosen. Once this is done the relation between the physical variables  $x$  and  $y$  and the parametric variables  $r$  and  $\theta$  will be found. This will give the surface displacement as a function of  $x$  and  $y$ .

# SOLUTION OF ORDINARY DIFFERENTIAL EQUATION FOR PARTICULAR BOTTOM TOPOGRAPHY

We shall consider the case where the function  $H$  of  $r$  is given by

$$H(r) = H_s + \delta(1 - r^2) \quad 0 \leq r \leq 1 \quad (84)$$

where  $H_s$  is the constant depth at the shore line and  $H_s + \delta$  is the depth of the deepest point of the lake. Upon inserting equation (84) into equations (81) and (83) and then inserting the results into equations (76 and (77), we obtain the following boundary value problem for  $\Omega_n$ :

$$r^2 \Omega_n'' + r \Omega_n' - [n^2 + 2i n \delta r^2] \Omega_n = \Gamma_n \quad \left\{ \begin{array}{l} n = 0, \pm 1, \pm 2, \dots \\ 0 \leq r \leq 1 \end{array} \right. \quad (85)$$

$$\left. \begin{array}{l} \Omega_n'(1) + i n [H_s - 1] \Omega_n(1) = \gamma_n \\ \lim_{r \rightarrow 0} \Omega_n(r) \text{ is finite} \end{array} \right\} \quad n = 0, \pm 1, \pm 2, \dots \quad (86)$$

Before obtaining the solution to this problem, we shall first show that

$$\int_0^1 \frac{1}{r} \Gamma_0(r) dr = \gamma_0 \quad (87)$$

To this end notice that equations (79) imply

$$\left. \begin{array}{l} \Gamma_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \sigma_x^{(1)} + \sigma_y^{(2)} \right] \left| \frac{dz}{dw} \right|^2 d\theta \\ \gamma_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \operatorname{Re} \sigma(1, \theta) \left( \frac{dz}{dw} \right)_{r=1}^* \right] d\theta \end{array} \right\} \quad (88)$$

Hence, it follows from the Cauchy-Riemann equations and equations (18) and (22) that

$$\begin{aligned}
\int_0^1 \frac{1}{r} \Gamma_0(r) dr &= \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} \left[ \sigma_{\mathbf{x}}^{(1)} + \sigma_{\mathbf{y}}^{(2)} \right] \left| \frac{dz}{dw} \right|^2 \frac{1}{r} dr d\theta \\
&= \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} \left[ \sigma_{\mathbf{x}}^{(1)} + \sigma_{\mathbf{y}}^{(2)} \right] \left( \frac{\partial \mathbf{x}}{\partial u} \frac{\partial \mathbf{y}}{\partial v} - \frac{\partial \mathbf{x}}{\partial v} \frac{\partial \mathbf{y}}{\partial u} \right) \frac{1}{r} dr d\theta \\
&= \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} \left[ \sigma_{\mathbf{x}}^{(1)} + \sigma_{\mathbf{y}}^{(2)} \right] \left( \frac{\partial \mathbf{x}}{\partial r} \frac{\partial \mathbf{y}}{\partial \theta} - \frac{\partial \mathbf{x}}{\partial \theta} \frac{\partial \mathbf{y}}{\partial r} \right) dr d\theta \\
&= \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} \left[ \sigma_{\mathbf{x}}^{(1)} + \sigma_{\mathbf{y}}^{(2)} \right] \frac{\partial(\mathbf{x}, \mathbf{y})}{\partial(r, \theta)} dr d\theta
\end{aligned}$$

Upon recalling that the unit circle in the  $\chi$ -plane transforms into the interior of the body of water in the  $z$ -plane, it follows from the theory of transformation of multiple integrals that

$$\int_0^1 \frac{1}{r} \Gamma_0(r) dr = \frac{1}{2\pi} \iint \left[ \sigma_{\mathbf{x}}^{(1)} + \sigma_{\mathbf{y}}^{(2)} \right] dx dy \quad (89)$$

where the integration is carried out over the entire surface of the region occupied by the body of water. Let  $S$  denote the distance measured along the boundary  $\mathcal{C}$  of this region. Then applying the Divergence Theorem to equation (89) yields

$$\int_0^1 \frac{1}{r} \Gamma_0(r) dr = \frac{1}{2\pi} \oint_{\mathcal{C}} \left[ \sigma^{(1)} n_1 + \sigma^{(2)} n_2 \right] dS$$

Hence, upon substituting equations (66) into this relation, we obtain

$$\int_0^1 \frac{1}{r} \Gamma_0(r) dr = \frac{1}{2\pi} \oint_{\mathcal{C}} \operatorname{Re} \left[ \sigma \left( \frac{dz}{dw} \right)^* \right] \left| \frac{dw}{dz} \right| dS \quad (90)$$

Now

$$dS = \sqrt{(dx)^2 + (dy)^2}$$

and

$$\left| \frac{dw}{dz} \right| = \sqrt{\left( \frac{dw}{dz} \right) \left( \frac{dw}{dz} \right)^*} = \sqrt{(u_x + iv_x)(u_x - iv_x)} = \sqrt{(u_x)^2 + (v_x)^2}$$

Hence,

$$\left| \frac{dw}{dz} \right| dS = \sqrt{[u_x dx]^2 + [v_x dy]^2 + [v_x dx]^2 + [u_x dy]^2}$$

Upon using the Cauchy-Riemann equations (11) and noting that  $u_x u_y = -v_y v_x$ , we find that

$$\left| \frac{dw}{dz} \right| dS = \sqrt{(u_x dx + u_y dy)^2 + (v_x dx + v_y dy)^2} = \sqrt{(du)^2 + (dv)^2}$$

Hence, it follows from equations (22)

$$\left| \frac{dw}{dz} \right| dS = \sqrt{\left( \frac{dr}{r} \right)^2 + (d\theta)^2}$$

But  $r$  is constant (and equal to unity) on  $\mathcal{C}$ . Hence,

$$\left| \frac{dw}{dz} \right| dS = d\theta$$

and equation (90) becomes

$$\int_0^1 \frac{1}{r} \Gamma_0(r) dr = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} \left[ \sigma(1, \theta) \left( \frac{dz}{dw} \right)^*_{r=1} \right] d\theta$$

Finally, comparing this with equation (88) shows that equation (87) holds.

Having established this result, we can now proceed to solve boundary value problem determined by equations (85) and (86) for  $n=0$ . In this case equations (85) and (86) become

$$r(r\Omega'_0)' = \Gamma_0 \quad 0 \leq r \leq 1 \quad (91)$$

$$\left. \begin{aligned} \Omega'_0(1) &= \gamma_0 \\ \lim_{r \rightarrow 0} \Omega_0(r) &\text{ is finite} \end{aligned} \right\} \quad (92)$$

Upon integrating equation (91) we get

$$r\Omega'_0 = l_1 + \int_0^r \Gamma_0(\rho) \frac{d\rho}{\rho}$$

where  $l_1$  is a constant of integration. Upon substituting this expression into the first boundary condition (eq. (92)), we find, upon using equation (87), that  $l_1 = 0$ . Therefore,

$$r\Omega'_0 = \int_0^r \Gamma_0(\rho) \frac{d\rho}{\rho}$$

Integrating this with respect to  $r$  yields

$$\Omega_0 = l_0 + \int_0^r \frac{1}{\tilde{r}} \int_0^{\tilde{r}} \frac{1}{\rho} \Gamma_0(\rho) d\rho d\tilde{r}$$

where  $l_0$  is a constant of integration. Upon integrating by parts, we find that

$$\Omega_0(r) = l_0 + \ln r \int_0^r \Gamma_0(\rho) \frac{d\rho}{\rho} + \int_r^1 \ln \rho \Gamma_0(\rho) \frac{d\rho}{\rho}$$

Hence, upon defining the function  $\mathcal{K}_0$  of  $r$  and  $\rho$  by

$$\mathcal{K}_0(r; \rho) = \begin{cases} \frac{1}{\pi} \ln r & \text{for } r \geq \rho \\ \frac{1}{\pi} \ln \rho & \text{for } \rho \geq r \end{cases} \quad (93)$$

the solution  $\Omega_0(r)$  can be written as

$$\Omega_0(r) = l_0 + \pi \int_0^1 \mathcal{K}_0(r; \rho) \Gamma_0(\rho) \frac{d\rho}{\rho} \quad (94)$$

Notice that this solution automatically satisfies the second boundary condition (92). This would not be the case if equation (87) did not hold; for then  $l_1$  would not vanish and the solution would therefore contain a term  $l_1 \ln r$  which is unbounded at the origin. On the other hand, the solution given by equation (94) is indeterminate to within an arbitrary constant  $l_0$ . But this is as it should be since the  $n = 0$  term in equation (75) is independent of  $\theta$ , and, as can be seen from equations (65) and (71), the original boundary value problem for  $\zeta$  involves only the partial derivatives of  $\zeta$  and hence determines  $\zeta$  only to within an arbitrary constant.

We now return to the boundary value problems determined by equations (85) and (86) and suppose that  $n \neq 0$ . Put

$$\left. \begin{aligned} \lambda_n &\equiv \sqrt{-2 \ln \delta} & n = \pm 1, \pm 2, \dots \\ r_n &\equiv \lambda_n r & n = \pm 1, \pm 2, \dots \end{aligned} \right\} \quad (95)$$

Then equation (85) becomes

$$r_n^2 \frac{d^2 \Omega_n}{dr_n^2} + r_n \frac{d\Omega_n}{dr_n} - (n^2 - r_n^2) \Omega_n = \Gamma_n \quad n = \pm 1, \pm 2, \dots$$

But this is just the nonhomogeneous Bessel's equation which has the solution

$$\Omega_n = \Omega_n^h + \Omega_n^p \quad (96)$$

where  $\Omega_n^h$  and  $\Omega_n^p$  are the homogeneous and particular solutions, respectively. The homogeneous solution is

$$\Omega_n^h = l_1^{(n)} J_n(r_n) + l_2^{(n)} Y_n(r_n) \quad n = \pm 1, \pm 2, \dots \quad (97)$$

where  $l_i^{(n)}$  for  $i = 1, 2$  are arbitrary constants and  $J_n$  and  $Y_n$  are the  $n^{\text{th}}$ -order

Bessel functions of the first and second kind, respectively. The Wronskian of  $J_n(r_n)$  and  $Y_n(r_n)$ , denoted by  $W(J_n, Y_n)$ , is

$$W(J_n, Y_n) \equiv \begin{vmatrix} J_n(r_n) & Y_n(r_n) \\ \frac{dJ_n(r_n)}{dr_n} & \frac{dY_n(r_n)}{dr_n} \end{vmatrix} = \frac{2}{\pi r_n}$$

Hence, a particular solution  $\Omega_n^p$  of nonhomogeneous equation (85) is (ref. 7, p. 155)

$$\Omega_n^p = \frac{\pi}{2} \left[ \int_0^r J_n(\lambda_n \rho) Y_n(\lambda_n r) \Gamma_n(\rho) \frac{d\rho}{\rho} + \int_r^1 J_n(\lambda_n r) Y_n(\lambda_n \rho) \Gamma_n(\rho) \frac{d\rho}{\rho} \right] \quad n = \pm 1, \pm 2, \dots \quad (98)$$

This solution is bounded. To show this, notice that (ref. 9, pp. 15 and 62) for  $r \rightarrow 0$

$$\left. \begin{aligned} J_n(\lambda_n r) &\sim \frac{\left(\pm \frac{1}{2} \lambda_n r\right)^{|n|}}{(|n|)!} \\ Y_n(\lambda_n r) &\sim \left(\pm \frac{1}{2} \lambda_n r\right)^{-|n|} \frac{1}{\pi} (|n| - 1)! \end{aligned} \right\} \quad n = \pm 1, \pm 2, \dots$$

Hence, by introducing these asymptotic forms for  $J_n$  and  $Y_n$  into equation (98), we find that<sup>4</sup>

$$\lim_{r \rightarrow 0} \Omega_n^p(r) = \frac{1}{|n|^2} \Gamma_n(0) \quad n = \pm 1, \pm 2, \dots$$

Thus,  $\Omega_n^p(r)$  and  $J_n(\lambda_n r)$  are bounded at  $r = 0$  and  $Y_n(\lambda_n r)$  is unbounded. Hence, equa-

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<sup>4</sup>It is not hard to verify that  $\Gamma_n(0) = 0$  for all  $n$ . Therefore,

$$\lim_{r \rightarrow 0} \Omega_n^p(r) = 0 \quad \text{for } n = \pm 1, \pm 2, \dots$$



tions (96) and (97) show that the solution  $\Omega_n$  is bounded if, and only if,  $l_2^{(n)} \equiv 0$ . The second boundary condition (86) can therefore only be satisfied if

$$l_2^{(n)} = 0 \quad \text{for } n = \pm 1, \pm 2, \dots \quad (99)$$

Upon combining equations (96) through (99) we get

$$\Omega_n(r) = l_1^{(n)} J_n(\lambda_n r) + \frac{\pi}{2} \left[ \int_0^r J_n(\lambda_n \rho) Y_n(\lambda_n r) \Gamma_n(\rho) \frac{d\rho}{\rho} + \int_r^1 J_n(\lambda_n r) Y_n(\lambda_n \rho) \Gamma_n(\rho) \frac{d\rho}{\rho} \right] \quad (100)$$

Hence

$$\begin{aligned} \Omega_n(1) &= l_1^{(n)} J_n(\lambda_n) + \frac{\pi}{2} Y_n(\lambda_n) \int_0^1 J_n(\lambda_n \rho) \Gamma_n(\rho) \frac{d\rho}{\rho} \\ \Omega_n'(1) &= l_1^{(n)} J_n'(\lambda_n) + \frac{\pi}{2} Y_n'(\lambda_n) \int_0^1 J_n(\lambda_n \rho) \Gamma_n(\rho) \frac{d\rho}{\rho} \end{aligned}$$

where  $J_n'(\lambda_n) \equiv \left[ (d/dr) J(\lambda_n r) \right]_{r=1}$  and so forth. Substituting these results into the first boundary condition (86) and using the recurrence relations for the Bessel functions to eliminate their derivatives gives

$$l_1^{(n)} = \frac{1}{\omega_n^{(1)}} \left[ \gamma_n - \omega_n^{(2)} \frac{\pi}{2} \int_0^1 J_n(\lambda_n \rho) \Gamma_n(\rho) \frac{d\rho}{\rho} \right] \quad (101)$$

where

$$\left. \begin{aligned} \omega_n^{(1)} &\equiv n \left[ 1 + i(H_S - 1) \right] J_n(\lambda_n) - \lambda_n J_{n+1}(\lambda_n) \\ \omega_n^{(2)} &\equiv n \left[ 1 + i(H_S - 1) \right] Y_n(\lambda_n) - \lambda_n Y_{n+1}(\lambda_n) \end{aligned} \right\} n = \pm 1, \pm 2, \dots \quad (102)$$

Inserting equation (101) into equation (100) shows that

$$\Omega_n(r) = \frac{\gamma_n}{\omega_n^{(1)}} J_n(\lambda_n r) + \frac{\pi}{2} \int_0^1 \mathcal{K}_n(r; \rho) \Gamma_n(\rho) \frac{d\rho}{\rho} \quad n = \pm 1, \pm 2, \dots \quad (103)$$

where we have put

$$\mathcal{K}_n(r; \rho) = \begin{cases} \left[ Y_n(\lambda_n r) - \frac{\omega_n^{(2)}}{\omega_n^{(1)}} J_n(\lambda_n r) \right] J_n(\lambda_n \rho) & \text{for } \rho \leq r \\ \left[ Y_n(\lambda_n \rho) - \frac{\omega_n^{(2)}}{\omega_n^{(1)}} J_n(\lambda_n \rho) \right] J_n(\lambda_n r) & \text{for } r \leq \rho \end{cases} \quad n = \pm 1, \pm 2, \dots \quad (104)$$

Since

$$J_{-n}(\lambda_{-n} r) = (-1)^n J_n(\lambda_{-n} r) = (-1)^n J_n(\lambda_n^* r) = (-1)^n [J_n(\lambda_n r)]^*$$

and

$$Y_{-n}(\lambda_{-n} r) = (-1)^n [Y_n(\lambda_n r)]^*$$

and since equations (79) show that

$$\Gamma_{-n} = \Gamma_n^*$$

$$\gamma_{-n} = \gamma_n^*$$

it is easy to see from equations (102) to (104) that

$$\omega_{-n}^{(j)} = (-1)^n [\omega_n^{(j)}]^* \quad \text{for } j = 1, 2$$

and hence that

$$\mathcal{K}_{-n}(r; \rho) = [\mathcal{K}_n(r; \rho)]^*$$

Therefore,

$$\Omega_{-n} = \Omega_n^* \quad \text{for } n = \pm 1, \pm 2, \dots \quad (105)$$

Upon substituting equations (103) and (94) into equation (75) and using equation (105), we find that the solution for the surface displacement is

$$\zeta(r, \theta) = l_0 + 2Re \sum_{n=1}^{\infty} \frac{\gamma_n}{\omega_n^{(1)}} J_n(\lambda_n r) e^{in\theta} + Re \pi \sum_{n=0}^{\infty} e^{in\theta} \int_0^1 \mathcal{K}_n(r; \rho) \Gamma_n(\rho) \frac{d\rho}{\rho} \quad (106)$$

where, upon collecting the definitions of the various quantities in this equation, we get

$$\mathcal{K}_0(r; \rho) = \begin{cases} \frac{1}{\pi} \ln r & \text{for } \rho \leq r \\ \frac{1}{\pi} \ln \rho & \text{for } r \leq \rho \end{cases} \quad (93)$$

$$\mathcal{K}_n(r; \rho) = \begin{cases} \left[ Y_n(\lambda_n r) - \frac{\omega_n^{(2)}}{\omega_n^{(1)}} J_n(\lambda_n r) \right] J_n(\lambda_n \rho) & \text{for } \rho \leq r \\ \left[ Y_n(\lambda_n \rho) - \frac{\omega_n^{(2)}}{\omega_n^{(1)}} J_n(\lambda_n \rho) \right] J_n(\lambda_n r) & \text{for } r \leq \rho \end{cases} \quad n = 1, 2, \dots \quad (104)$$

$$\left. \begin{aligned} \omega_n^{(1)} &= n \left[ 1 + i(H_S - 1) \right] J_n(\lambda_n) - \lambda_n J_{n+1}(\lambda_n) \\ \omega_n^{(2)} &= n \left[ 1 + i(H_S - 1) \right] Y_n(\lambda_n) - \lambda_n Y_{n+1}(\lambda_n) \end{aligned} \right\} \quad n = 1, 2, \dots \quad (102)$$

$$\lambda_n = \sqrt{-2in\delta} = e^{3\pi i/4} \sqrt{2n\delta} = (i - 1) \sqrt{n\delta} \quad (95)$$

$$\left. \begin{aligned} \Gamma_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \left[ \sigma_x^{(1)} + \sigma_y^{(2)} \right] \left| \frac{dz}{dw} \right|^2 d\theta \\ \gamma_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \operatorname{Re} \left[ \sigma(1, \theta) \left( \frac{dz}{dw} \right)_{r=1}^* \right] d\theta \end{aligned} \right\} \quad n = 0, 1, 2, \dots \quad (79)$$

$$\sigma \equiv \sigma^{(1)} + i\sigma^{(2)} \quad (67)$$

Finally, upon inserting equation (49) into equation (47), we find that

$$\left. \begin{aligned} \sigma^{(1)} &= \tau_1^\dagger e^{-H/2} 2 \sin \frac{H}{2} + \tau_2^\dagger \left( 1 - e^{-H/2} 2 \cos \frac{H}{2} \right) \\ \sigma^{(2)} &= \tau_1^\dagger \left( e^{-H/2} 2 \cos \frac{H}{2} - 1 \right) + \tau_2^\dagger e^{-H/2} 2 \sin \frac{H}{2} \end{aligned} \right\} \quad (107)$$

in which

$$H(r) = H_s + \delta(1 - r^2) \quad 0 \leq r \leq 1 \quad (84)$$

and where we have put

$$\tau_i^\dagger \equiv \frac{2\pi\tau_i}{gd} \quad \text{for } i = 1, 2 \quad (108)$$

If  $h/d$  is larger than 1, then equations (107) may be replaced by the less exact relations

$$\left. \begin{aligned} \sigma^{(1)} &\approx \tau_2^\dagger \\ \sigma^{(2)} &\approx -\tau_1^\dagger \end{aligned} \right\} \quad \text{for } \frac{h}{d} > 1 \quad (48)$$

Notice that when these latter approximate relations are used for  $\sigma^{(1)}$  and  $\sigma^{(2)}$  and when the components of the components of the wind stress  $\tau_1$  and  $\tau_2$  are constants, then (see eq. (79))

$$\Gamma_n = 0 \quad \text{for } n = 0, 1, 2, \dots$$

and only the first summation appears in solution (106).

In order to evaluate the integrals in equation (79), it is necessary to know the relation between the physical coordinates  $x$  and  $y$  and the polar coordinates  $r$  and  $\theta$  in the  $\chi$ -plane, and also to know  $dz/dw = \chi(dz/d\chi)$  as a function of  $\chi$  (or equivalently as a function of  $r$  and  $\theta$ ). These relations can be determined as soon as the function

$$z \rightarrow \chi$$

which maps the interior of the region occupied by the body of water in the physical plane into the interior of the unit circle in the  $\chi$ -plane is selected.

## SOLUTION OF COMPLETE BOUNDARY VALUE PROBLEM FOR ELLIPTICAL LAKE OR SEA BY CONFORMAL MAPPING

In order to find the mapping  $z \rightarrow \chi$ , the specific shape of the body of water must be chosen. We shall, therefore, suppose that the body of water has the shape of an ellipse whose semimajor axis has length  $L_1$  and whose semiminor axis has length  $L_2$  as shown in figure 6. In addition, we shall assume that the body of water is deepest at its center. Since the depth-distribution function (eq. (84)) has its maximum at the center of

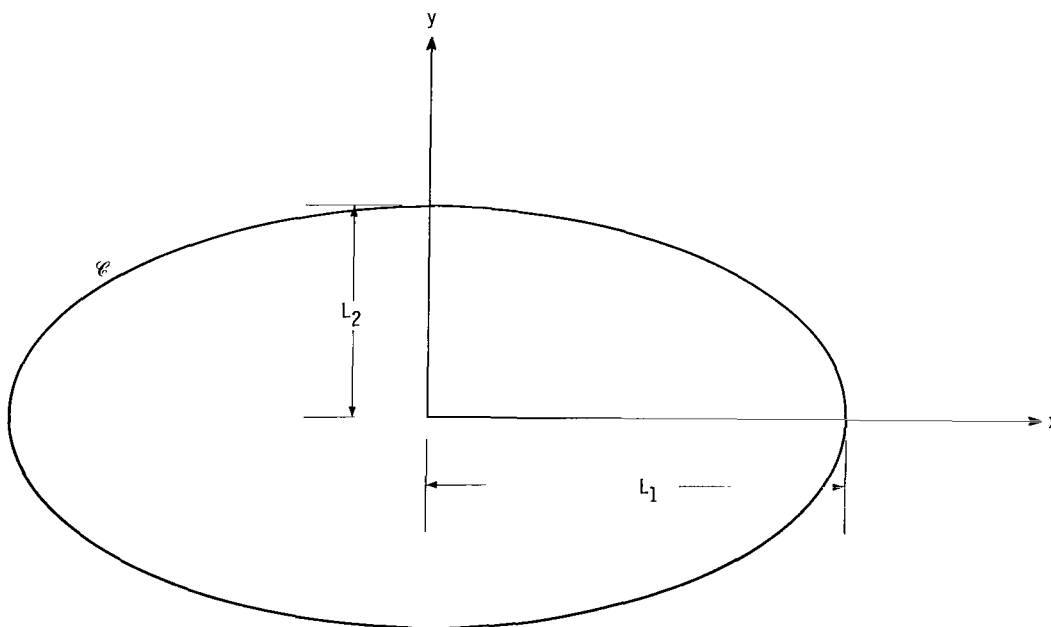


Figure 6. - Shape of body of water.

the unit circle, the mapping  $z \rightarrow \chi$  must be chosen to map the center of the ellipse into the center of the unit circle. For convenience the mapping will be chosen so that it maps the positive real axis in the  $z$ -plane into the positive real axis in the  $\chi$ -plane. As has been shown in the preceding discussion, these conditions are sufficient to completely determine the mapping

$$z \rightarrow \chi$$

In order to introduce this mapping it is convenient to introduce the parametric complex variable

$$t = \xi + i\eta \quad (109)$$

The  $t$ -plane is shown in figure 7. Let

$$K = K(k) \quad (110)$$

be the complete elliptic integral of the first kind of modulus  $k$  and let

$$K' = K(k') \quad (111)$$

where

$$k' = \sqrt{1 - k^2} \quad (112)$$

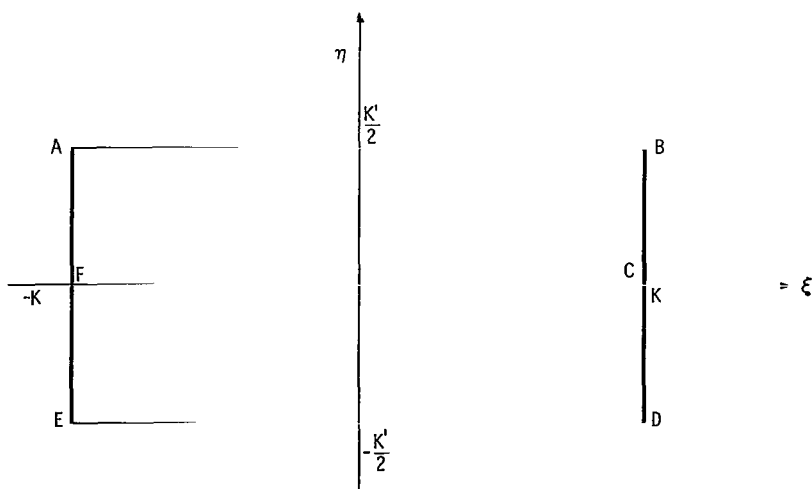


Figure 7. -  $t$ -plane.

is the complementary modulus. Then the mapping

$$z = I \sin \frac{\pi t}{2K} \quad (113)$$

where

$$I \equiv \sqrt{L_1^2 - L_2^2} \quad (114)$$

transforms the rectangular region in the  $t$ -plane shown in figure 7 into the elliptical re-

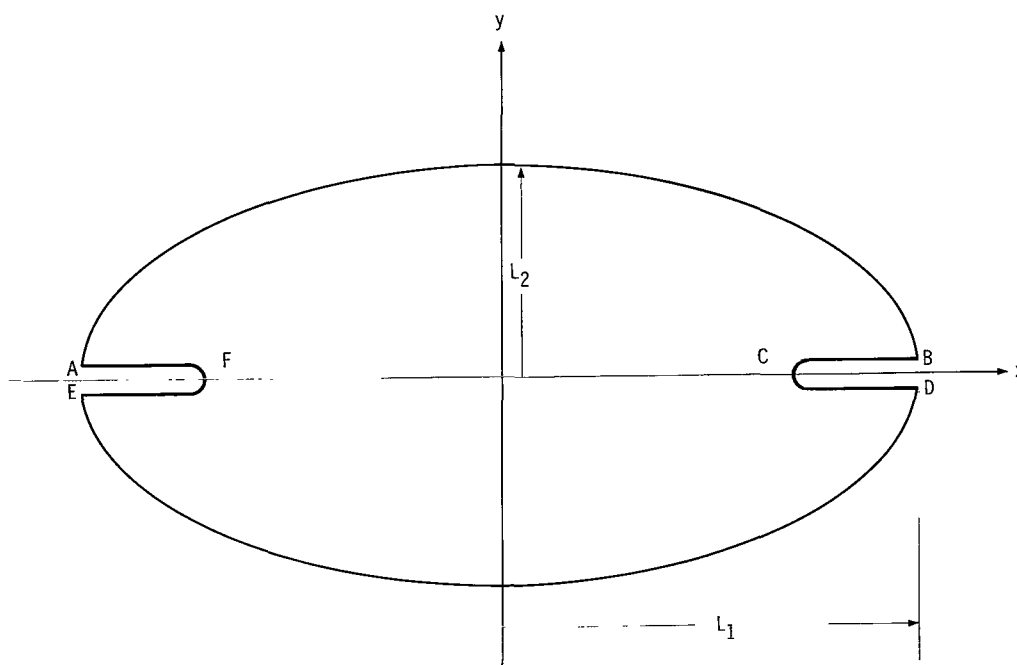


Figure 8. -  $z$ -Plane.

gion (with two slits) in the  $z$ -plane shown in figure 8. The mapping

$$\chi = k^{1/2} \operatorname{sn}(t, k) \quad (115)$$

transforms the rectangular region in the  $t$ -plane shown in figure 7 into the interior of the unit circle with two slits in the  $\chi$ -plane shown in figure 9. Hence, the combination

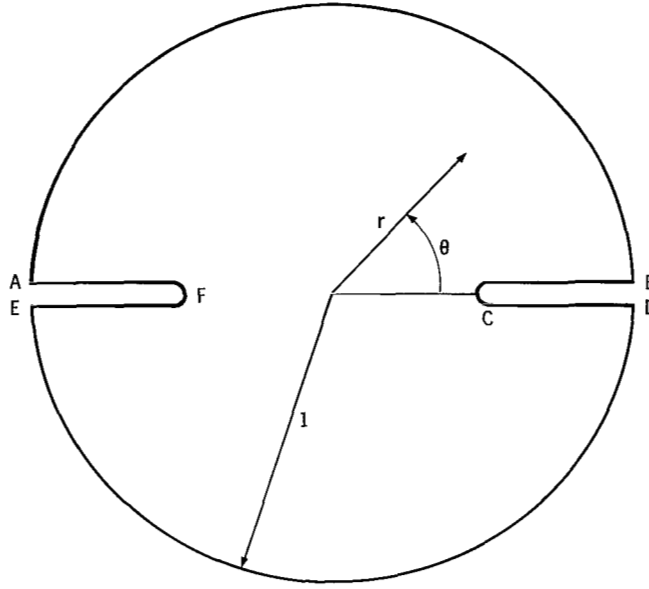


Figure 9. -  $\chi$ -plane.

of mapping (115) with the inverse of mapping (113) is the mapping (ref. 10, p. 371)

$$\chi = k^{1/2} \operatorname{sn} \left[ \frac{2K}{\pi} \sin^{-1} \left( \frac{z}{I} \right), k \right] \quad (116)$$

which transforms the elliptical region with two slits in the  $z$ -plane (shown in fig. 8) into the interior of the unit circle with two slits in the  $\chi$ -plane (shown in fig. 9). Since each cut in the  $z$ -plane corresponds precisely to a cut in the  $\chi$ -plane, these cuts can be sealed and hence, mapping (116) is the one with the desired properties. However, it is more convenient to deal with this mapping in terms of its components (mappings (113) and (115)). It is easy to verify that the modulus  $k$  is determined in terms of the physical dimensions of the ellipse by Jacobi's nome (ref. 10, p. 371)

$$e^{-\pi K'/K} = \left( \frac{L_1 - L_2}{L_1 + L_2} \right)^2 \quad (117)$$

Upon taking the real and imaginary parts of equation (113), we find that the physical variables  $x$  and  $y$  are given in terms of the parametric variables  $\xi$  and  $\eta$  by



$$\left. \begin{aligned} x &= I \sin \left( \frac{\pi}{2K} \xi \right) \cosh \left( \frac{\pi}{2K} \eta \right) \\ y &= I \cos \left( \frac{\pi}{2K} \xi \right) \sinh \left( \frac{\pi}{2K} \eta \right) \end{aligned} \right\} \quad (118)$$

Upon taking the real and imaginary parts of equation (115) and recalling that

$$re^{i\theta} \equiv \chi$$

we find that the polar coordinates  $r$  and  $\theta$  in the  $\chi$ -plane can be expressed in terms of the parametric variables  $\xi$  and  $\eta$  by (ref. 11, p. 24)

$$\left. \begin{aligned} \tan \theta &= \left[ \frac{\operatorname{cn}(\xi, k) \operatorname{dn}(\xi, k)}{\operatorname{sn}(\xi, k)} \right] \left[ \frac{\operatorname{sn}(\eta, k') \operatorname{cn}(\eta, k')}{\operatorname{dn}(\eta, k')} \right] \\ r \cos \theta &= \sqrt{k} \left[ \frac{\operatorname{sn}(\xi, k) \operatorname{dn}(\eta, k')}{1 - \operatorname{sn}^2(\eta, k') \operatorname{dn}^2(\xi, k)} \right] \end{aligned} \right\} \quad (119)$$

These equations now determine the physical coordinates  $x$  and  $y$  parametrically (with parameters  $\xi$  and  $\eta$ ) in terms of the polar coordinates  $r$  and  $\theta$  in the  $\chi$ -plane. Actually it is possible to eliminate the parametric variables  $\xi$  and  $\eta$  between equations (118) and (119) and obtain an expression for  $x$  and  $y$  directly in terms of  $r$  and  $\theta$  but we shall not do this here.

Finally, in order to evaluate the integrals in equations (79), it is necessary to obtain an expression for  $dz/dw$ . Because equation (20) implies

$$\frac{dz}{dw} = \chi \frac{dz}{d\chi}$$

it follows that

$$\frac{dz}{dw} = \chi \frac{dz}{dt} \frac{dt}{d\chi}$$

Upon differentiating equations (113) and (115), we find that

$$\frac{dz}{dt} = \frac{I\pi}{2K} \cos \left( \frac{\pi}{2K} t \right)$$

and

$$\frac{dt}{d\chi} = \frac{1}{k^{1/2} \text{cn}(t, k) \text{dn}(t, k)}$$

Therefore,

$$\frac{dz}{dw} = \frac{\pi I \chi}{2 \sqrt{k} K} \frac{\cos\left(\frac{\pi}{2K} t\right)}{\text{cn}(t, k) \text{dn}(t, k)} \quad (120)$$

In order to separate the real and imaginary parts of the denominator of this expression, notice that the addition theorems for elliptic functions (ref. 11, p. 23) show that

$$\text{sn}(t + t^*) - \text{sn}(t - t^*) = \frac{2 \text{sn } t^* \text{cn } t \text{dn } t}{1 - k^2 \text{sn}^2 t \text{sn}^2 t^*}$$

Hence, because  $\text{sn}(t^*) = (\text{sn } t)^*$ , we see that

$$\text{sn}(2\xi) - \text{sn}(2i\eta) = \frac{2(\text{sn } t)^* \text{cn } t \text{dn } t}{1 - k^2 |\text{sn } t|^4}$$

It now follows from equation (115) that

$$\begin{aligned} \frac{1}{\text{cn } t \text{dn } t} &= \frac{2}{\sqrt{k}} \frac{\chi^*}{\left(1 - |\chi|^4\right) \left[\text{sn}(2\xi, k) - \text{sn}(2i\eta, k)\right]} \\ &= \frac{2}{\sqrt{k}} \frac{\chi^*}{\left(1 - |\chi|^4\right)} \frac{1}{\text{sn}(2\xi, k) - i \text{tn}(2\eta, k')} \end{aligned}$$

(In more recent references the symbol  $sc$  is used instead of the symbol  $tn$  used here.) Substituting this into equation (120) and using equation (21) yields

$$\frac{dz}{dw} = \frac{\pi I}{kK} \frac{r^2}{1 - r^4} \frac{\cos\left(\frac{\pi}{2K} t\right)}{\text{sn}(2\xi, k) - i \text{tn}(2\eta, k')}$$

or

$$\frac{dz}{dw} = \frac{\pi I}{kK} \left( \frac{r^2}{1-r^4} \right) \frac{\cos\left(\frac{\pi}{2K}\xi\right)\cosh\left(\frac{\pi}{2K}\eta\right) - i \sin\left(\frac{\pi}{2K}\xi\right)\sinh\left(\frac{\pi}{2K}\eta\right)}{\operatorname{sn}(2\xi, k) - i \operatorname{tn}(2\eta, k')} \quad 0 \leq r < 1, -\pi \leq \theta \leq \pi \quad (121)$$

This expression is indeterminate on the boundary of the unit circle ( $r = 1$ ). It is, therefore, necessary to obtain an alternate form for this expression on the boundary. To this end notice that figure 7 shows, at  $r = 1$ ,  $\eta = \pm(K'/2)$ .

Now it follows from the addition theorems for the elliptic functions that

$$\operatorname{cn}\left(\xi \pm i \frac{K'}{2}\right) \operatorname{dn}\left(\xi \pm i \frac{K'}{2}\right) = \frac{1+k}{\sqrt{k}} \frac{\operatorname{cn} \xi \operatorname{dn} \xi - k \operatorname{sn}^2 \xi \operatorname{cn} \xi \operatorname{dn} \xi \mp i(\operatorname{sn} \xi \operatorname{dn}^2 \xi + k \operatorname{cn}^2 \xi \operatorname{sn} \xi)}{(1 + k \operatorname{sn}^2 \xi)^2}$$

Since (ref. 11, p. 20)

$$\operatorname{dn}^2 \xi = 1 - k^2 \operatorname{sn}^2 \xi$$

and

$$\operatorname{cn}^2 \xi = 1 - \operatorname{sn}^2 \xi$$

this becomes

$$\operatorname{cn}\left(\xi \pm i \frac{K'}{2}\right) \operatorname{dn}\left(\xi \pm i \frac{K'}{2}\right) = \frac{1+k}{\sqrt{k}} \left( \frac{1 - k \operatorname{sn}^2 \xi}{1 + k \operatorname{sn}^2 \xi} \right) \left[ \frac{\operatorname{cn} \xi \operatorname{dn} \xi}{1 + k \operatorname{sn}^2 \xi} \mp (1+k)i \frac{\operatorname{sn} \xi}{1 + k \operatorname{sn}^2 \xi} \right]$$

However, equations (119) implies that at  $r = 1$  (or  $\eta = \pm(1/2)K'$ )

$$\left. \begin{aligned} \cos \theta &= \frac{(1+k)\operatorname{sn} \xi}{1 + k \operatorname{sn}^2 \xi} \\ \sin \theta &= \pm \frac{\operatorname{cn} \xi \operatorname{dn} \xi}{1 + k \operatorname{sn}^2 \xi} \end{aligned} \right\} \quad (122)$$

Hence,

$$\operatorname{cn}\left(\xi \pm i \frac{K'}{2}\right) \operatorname{dn}\left(\xi \pm i \frac{K'}{2}\right) = \pm \frac{1+k}{i\sqrt{k}} \left( \frac{1 - k \operatorname{sn}^2 \xi}{1 + k \operatorname{sn}^2 \xi} \right) e^{i\theta}$$

Upon inserting this into equation (120) evaluated at  $r = 1$ , we find that

$$\begin{aligned} \left. \frac{dz}{dw} \right|_{r=1} &= \pm i \frac{\pi I}{2(1+k)K} \left( \frac{1 + k \operatorname{sn}^2 \xi}{1 - k \operatorname{sn}^2 \xi} \right) \cos \left( \frac{\pi}{2K} \xi \pm i \frac{\pi K'}{4K} \right) \\ &= \frac{\pi I}{2(1+k)K} \left( \frac{1 + k \operatorname{sn}^2 \xi}{1 - k \operatorname{sn}^2 \xi} \right) \left[ \sin \left( \frac{\pi}{2K} \xi \right) \sinh \left( \frac{\pi K'}{4K} \right) \pm i \cos \left( \frac{\pi}{2K} \xi \right) \cosh \left( \frac{\pi K'}{4K} \right) \right] \end{aligned}$$

But it follows from equations (117)<sup>5</sup> and the use of the exponential forms of the hyperbolic functions that

$$\sinh \left( \frac{\pi K'}{4K} \right) = \frac{L_2}{I}$$

and

$$\cosh \left( \frac{\pi K'}{4K} \right) = \frac{L_1}{I}$$

---

<sup>5</sup>Note that

$$\exp \left( -\frac{\pi K'}{4K} \right) = \sqrt{\frac{L_1 - L_2}{L_1 + L_2}}$$

and

$$\exp \left( \frac{\pi K'}{4K} \right) = \sqrt{\frac{L_1 + L_2}{L_1 - L_2}}$$

Hence,

$$\left. \frac{dz}{dw} \right|_{r=1} = \frac{\pi}{2(1+k)K} \left( \frac{1+k \operatorname{sn}^2 \xi}{1-k \operatorname{sn}^2 \xi} \right) \left[ L_2 \sin \left( \frac{\pi \xi}{2K} \right) \pm i L_1 \cos \left( \frac{\pi \xi}{2K} \right) \right] \quad (123)$$

where the plus sign corresponds to  $\theta \geq 0$  and the minus sign is used when  $\theta \leq 0$ . Since equation (123) is to be used in the integral of the second equation (79) and since the integration here is with respect to  $\theta$ , it is convenient to have an expression for the variable  $\xi$  as a function of  $\theta$ . This can be accomplished by solving equation (122) for  $\operatorname{sn} \xi$  as a function of  $\theta$  to obtain

$$\operatorname{sn} \xi = \frac{1+k \pm \sqrt{1+k^2 - 2K \cos 2\theta}}{2k \cos \theta}$$

Since  $\xi = K$  at  $\theta = 0$ , it is easy to see that the minus sign must hold in this equation and therefore that

$$\xi = \operatorname{sn}^{-1} \left[ \frac{1+k - \sqrt{1+k^2 - 2k \cos 2\theta}}{2k \cos \theta}, k \right] \quad \text{for } r = 1 \quad (124)$$

Since  $\eta = \pm(K'/2)$  when  $r = 1$ , it follows from equation (118) and the relations immediately preceding equation (123) that

$$\left. \begin{aligned} x &= L_1 \sin \left( \frac{\pi}{2K} \xi \right) \\ y &= \pm L_2 \cos \left( \frac{\pi}{2K} \xi \right) \end{aligned} \right\} \quad \text{for } r = 1 \quad (125)$$

Thus when the wind stress components  $\tau_1$  and  $\tau_2$  are given as functions of  $x$  and  $y$ , the integral in the second equation (79) can easily be evaluated by using equations (67), (84), (107), (108), and (123) to (125). Similarly the integral in the first equation (79) can be evaluated by using equations (84), (107), (108), (118), (119), and (121). These results can then be substituted into equation (106) to determine the surface displacement  $\zeta$  as a function of  $r$  and  $\theta$ . Equations (118) and (119) will then give  $\zeta$  as a function of  $x$  and  $y$  and this completes the solution to the problem.

Since the geometry chosen does not represent any particular lake or sea, it will not

be worthwhile to carry out a complete parametric study for various functional forms of the wind stress. Instead we shall only obtain numerical results for the simplest case which illustrates all the features involved. Thus, we shall suppose that  $h/d > 1$  and that approximate equation (49) can be used for  $\sigma^{(1)}$  and  $\sigma^{(2)}$ . We shall also suppose that the wind stress is constant and is given in dimensionless form by

$$\left. \begin{aligned} \tau_1^\dagger &= \tau_0^\dagger \cos \vartheta \\ \tau_2^\dagger &= \tau_0^\dagger \sin \vartheta \end{aligned} \right\} \quad (126)$$

where  $\tau_0^\dagger$  is the constant dimensionless magnitude of the wind-stress and  $\vartheta$  is the angle between the wind stress and the  $x$ -axis. Then in this case, as indicated previously,  $\Gamma_n = 0$  for  $n = 0, 1, 2, \dots$  and it follows from equations (48), (67), (79), (106), (108), (123), and (124) that the solution for the surface displacement is

$$\frac{\zeta - \zeta_0}{\tau_0^\dagger L_1} = \frac{1}{2(1+k)K} \operatorname{Re} \sum_{n=1}^{\infty} \frac{1}{\omega_n^{(1)}} J_n(\lambda_n r) e^{in\theta} \mu_n \quad (127)$$

where

$$\left. \begin{aligned} \mu_n &\equiv \int_{-\pi}^{\pi} e^{-in\theta} \frac{1+k \operatorname{sn}^2 \xi}{1-k \operatorname{sn}^2 \xi} \left[ \frac{L_2}{L_1} \sin\left(\frac{\pi}{2K} \xi\right) \sin \vartheta - \frac{\theta}{|\theta|} \cos\left(\frac{\pi}{2K} \xi\right) \sin \vartheta \right] d\theta \\ \xi &= \operatorname{sn}^{-1} \left( \frac{1+k - \sqrt{1+k^2 - 2k \cos 2\theta}}{2k \cos \theta}, k \right) \end{aligned} \right\} \quad (128)$$

## DISCUSSION

A method for obtaining solutions to a certain class of partial differential equations is introduced. This class of equation includes Welander's equation for the wind-driven circulation in shallow seas and lakes for a large class of bottom topographies. The technique, together with some results based on conformal mapping is used to reduce the prob-

lem of finding solutions to Welander's equation for a closed body of water to the problem of solving an ordinary differential equation which depends on a parameter. An approximate form of this ordinary differential equation which is valid when the depth of the body of water is larger than one half of the Eckman depth is solved for a particular bottom topography. This solution takes on a particularly simple form when the wind stress is constant and the depth of the lake is greater than the Eckman depth.

The following discussion concerns the application of this solution to an elliptically shaped body of water with an approximately paraboloidal bottom topography whose depth is larger than the Eckman depth and for which the wind stress is constant. No attempt is made to conduct a parametric study to demonstrate the effect of the governing dimensionless parameters since this would be of practical value only if a particular lake or sea were being studied. For this reason numerical results will only be given for the surface displacement of the surface of the body of water. All other flow quantities of interest can easily be expressed in terms of this quantity by using the results given in reference 1.

The surface displacement of the body of water is given in terms of the intermediate variables  $r$  and  $\theta$  by equations (127) and (128) where the modulus  $k$  of the complete elliptic integrals is determined from equation (117) and the parameters  $\omega_n^{(1)}$  and  $\lambda_n$  are given by equations (102) and (95), respectively. Since the physical coordinates  $x$  and  $y$  of the horizontal plane of the body of water are related to the intermediate variables  $r$  and  $\theta$  parametrically by equations (118) and (119), the surface displacement is known as a function of  $x$  and  $y$ . These results involve the physical parameters  $L_2/L_1$ ,  $H_s$ ,  $\delta$ , and  $\vartheta$ . The first three of these parameters characterize the geometry of the body of water and the last one is the direction of the wind stress. All the numerical calculations were performed for  $H_s = 5$  and  $\delta = 4$ . The results are shown as isometric projections. The results of the calculations for  $\vartheta = 0$  (i. e., when the wind direction is along the major axis of the ellipse) are shown in figure 10. The figure shows that the water piles up downwind and the surface is s-shaped across the wind direction. The results for  $\vartheta = 45^\circ$  and  $\vartheta = 90^\circ$  are shown in figures 11 and 12, respectively. In all three figures the ratio  $L_2/L_1$  of the semiminor to the semimajor axis of the ellipse is taken as  $2/3$ . The effect of fixing  $\vartheta$  at  $45^\circ$  and changing  $L_2/L_1$  to  $1/2$  is shown in figure 13. It can be seen from these figures that the amount of surface displacement is essentially determined by the distance across the lake in the direction of the wind. The greater this distance, the larger the surface displacement.

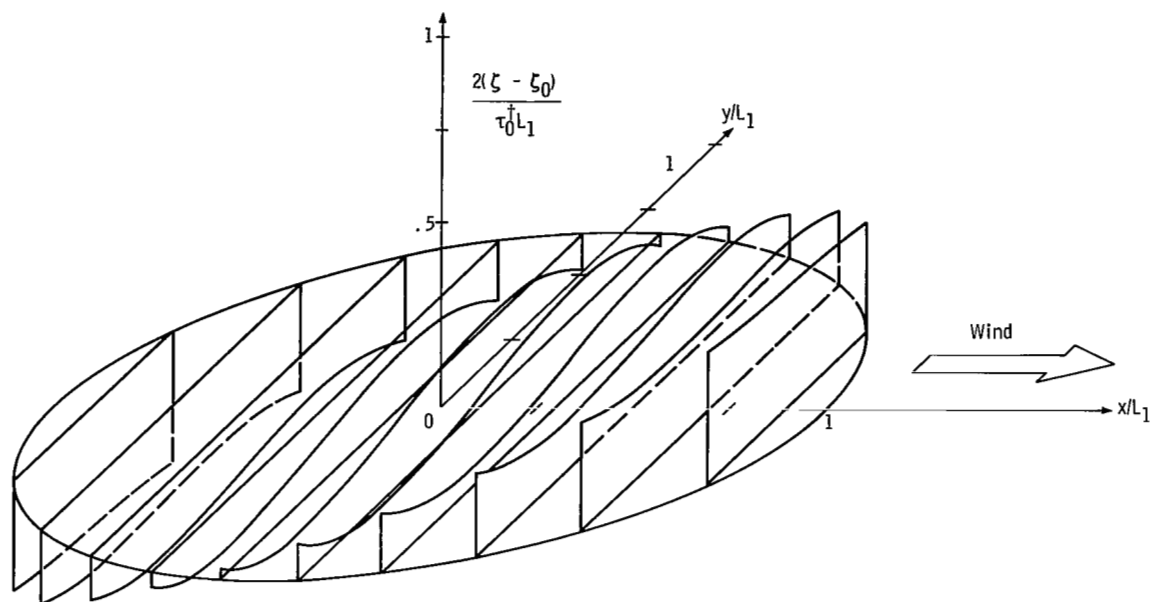


Figure 10. - Surface displacement of an elliptic lake. Ratio of minor to major axis,  $L_2/L_1 = 2/3$ ; wind parallel to major axis ( $\phi = 0$ );  $\delta = 4$ ;  $H_S = 5$ .

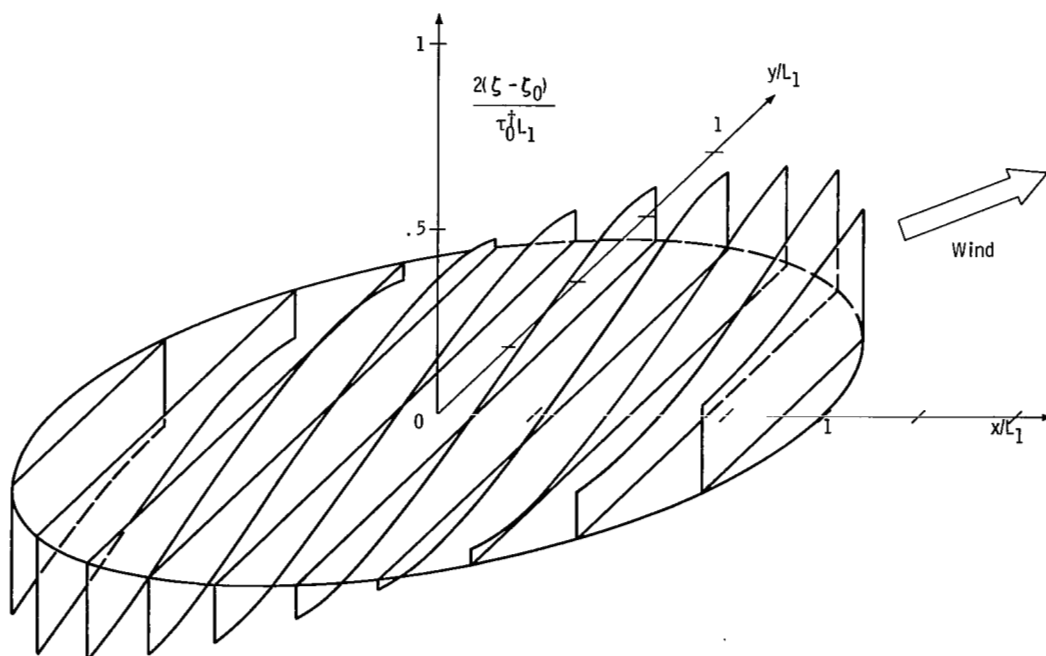


Figure 11. - Surface displacement of an elliptic lake. Ratio of minor to major axis,  $L_2/L_1 = 2/3$ ; wind at  $45^\circ$  to axes ( $\phi = 45^\circ$ );  $\delta = 4$ ;  $H_S = 5$ .



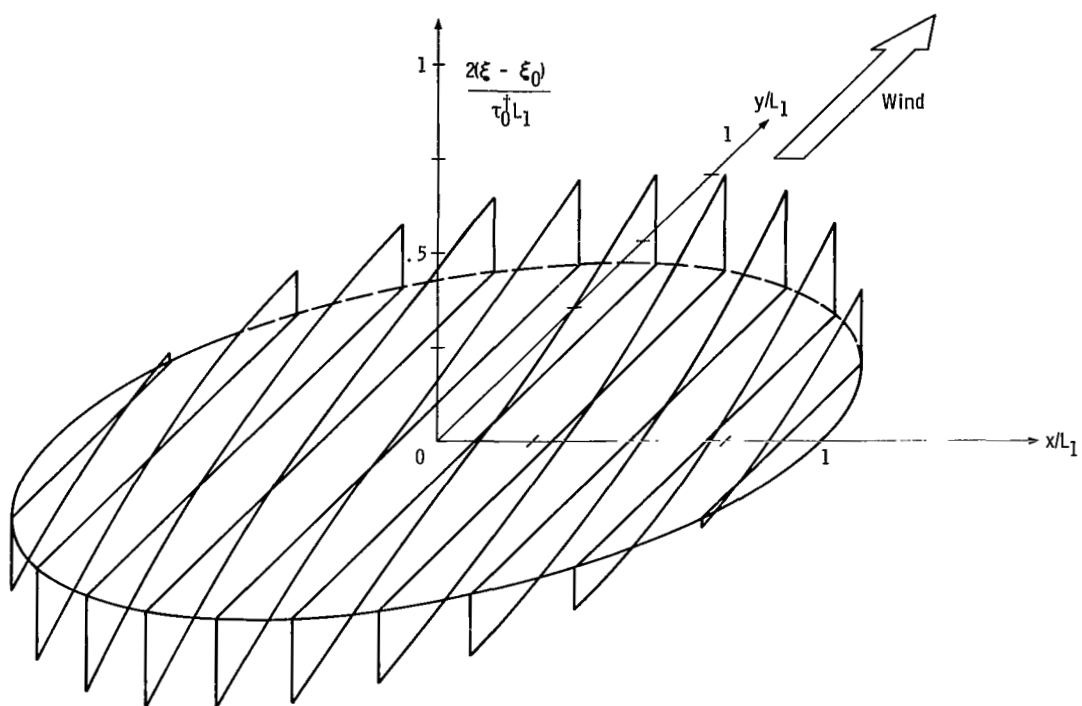


Figure 12. - Surface displacement of an elliptic lake. Ratio of minor to major axis,  $L_2/L_1 = 2/3$ ; wind parallel to minor axis ( $\vartheta = 90^\circ$ );  $\delta = 4$ ;  $H_S = 5$ .

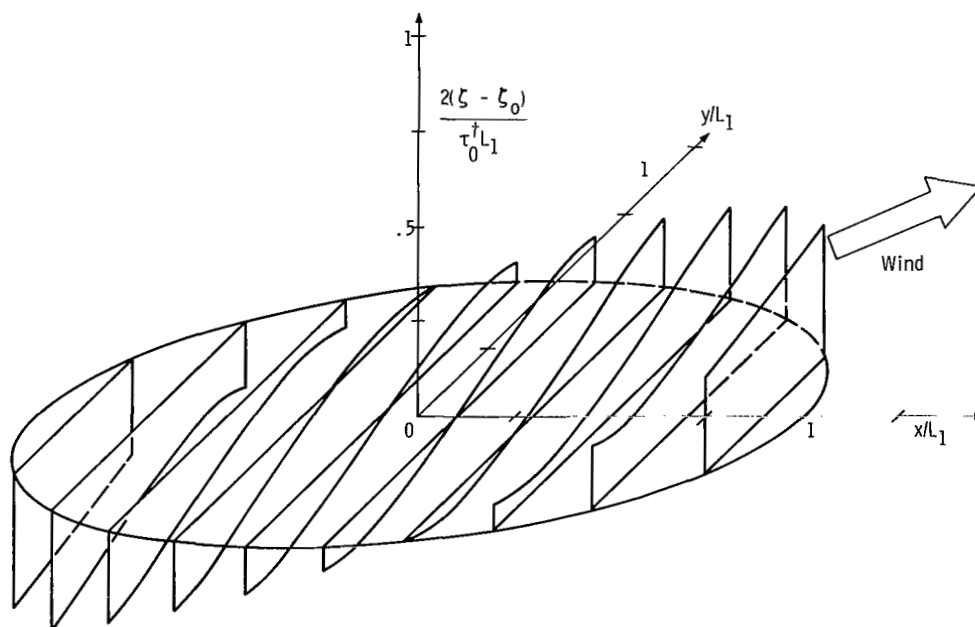


Figure 13. - Surface displacement of an elliptic lake. Ratio of minor to major axis,  $L_2/L_1 = 1/2$ ; wind at  $45^\circ$  to axes ( $\vartheta = 45^\circ$ );  $\delta = 4$ ;  $H_S = 5$ .

## CONCLUSIONS

A technique for obtaining solutions to a certain class of partial differential equations is given. This technique is applied to the equation which describes the wind-driven circulation in a shallow sea or lake. It is shown that for a large class of depth distributions this technique can be used to reduce this partial differential equation to an ordinary differential equation. Complete analytical solutions are obtained for a specific geometry of the lake or sea.

Lewis Research Center,  
National Aeronautics and Space Administration,  
Cleveland, Ohio, June 15, 1970,  
129-01.

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